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THE ELEMENTS OF *EUCLID*;

With Select THEOREMS out of *ARCHIMEDES*.

By the Learned ANDREW TACQUET.

To which are added,

Practical COROLLARIES, shewing the Uses of many of the Propositions.

The Whole abridg'd, and Publish'd in *English*,
By WILLIAM WHISTON, M. A.
Mr. *Lucas*'s Professor of the *Mathematicks* in the University of *Cambridge*.

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AN
Hiſtorical Account
OF THE
RISE and PROGRESS
OF THE
MATHEMATICKS.

IT ſeem'd meet to me when I was about to ſet forth the Elements of the Mathematicks, to premiſe a few Things concerning the Riſe and Excellency of this Science, that its Candidates may underſtand what a Kind of Science it is to which they are about to dedicate themſelves; and that it may be made manifeſt againſt thoſe who ſlight thoſe Things whereof they are ignorant, of how great Value and Dignity

this Knowledge is, which the wiſeſt Men of all Ages have, with incredible Study, labour'd to attain unto, and become poſſeſs'd of. Moreover, I muſt own that *Peter Ramus*'s Labours have been of great Service to me in the compiling of this Account, who in the whole firſt Book of his Inſtitution, which is not a little one, hath out of *Proclus*, *Laertius*, *Gellius*, *Polybius*, *Tzetzes*, and others, compoſed a Mathematical Hiſtory both accurately and copiouſly.

The Mathematical Sciences were the firſt of all other amongſt Men, if we may believe *Joſephus*. He, *Book* I. *Chap.* 3. writeth, that the Poſterity of *Seth* obſerved the Order of the Heavens, and the Courſes of the Stars. And leſt theſe Inventions ſhould ſlip out of the Knowledge of Men, *Adam* having predicted a twofold Deſtruction of the Earth, one by a Deluge, the other by Fire, they rais'd two Columns, one of Bricks, of Stone the other; and inſcribed their Inventions upon them, that if the Brick one ſhould happen to be deſtroy'd by the Deluge, that of Stone, which would remain, might afford Men an Opportunity of being inſtructed, and preſent to their View the Things which it had inſcrib'd on it. They ſay alſo, that that ſtone Pillar, which even in our Days is ſeen in

in *Syria*, was dedicated by them. This *Josephus* says: whom I leave to vouch for the Story.

That the *Assyrians* and *Chaldeans* were the first after the Flood, who applied themselves to the Mathematicks, is delivered by the same *Josephus*; as also by *Pliny*, *Diodorus*, and *Cicero*. But the Mathematick Arts, which first sprang amongst the *Chaldeans*, amongst whom they flourished, were afterwards transferr'd out of *Chaldea* and *Assyria* unto the *Egyptians*, by *Abraham*. For, when, at the Command of God, he went forth from his native Soil into *Palestine*, and from thence into *Egypt*, and perceiv'd the *Egyptians* to be taken with the Study of good Arts, and to be of a remarkable Disposition and Capacity for Learning, (as *Josephus* testifies, *Book* I. *Chap.* 9.) he communicated to them Arithmetick and Astronomy; and consequently Geometry, which must of Necessity go before Astronomy. In which Studies afterwards the *Egyptians* so flourish'd, that *Aristotle*, 1 *Metaph. Chap.* 1. doth affirm, That the Mathematick Arts were first found out in *Egypt*, by their Priests; who by their Employments were at leisure for these Things.

Then these Arts crossing the Sea out of *Egypt*, came to the Philosophers of *Greece*: For *Thales* the *Milesian*, who

 flourish'd

flourish'd 584 Years before Christ, wa the first of the *Greeks*, who coming in to *Egypt*, transferr'd Geometry from thenc into *Greece*. He it was indeed, who, besides other Things, found out the 5th 15th, and 26th Propositions of the first Book. To the same are also owing the 2d, 3d, 4th, 5th, of the fourth Book The same Person began to observe the Equinoxes and Solstices, as *Laertius* testifies; and he was the first who foretold an Eclipse of the Sun, as *Hippias* and *Aristotle* write; and *Tzetzes* saith, That he also foretold an Eclipse of the Moon to King *Cyrus*. For which Things sake he is to be look'd on as the first Founder and Author of the Mathematical Sciences in *Greece*.

After him was *Pythagoras* of *Samos*: Which most ancient Philosopher, exceedingly improv'd and adorn'd the Mathematick Sciences. And he so gave himself to Arithmetick in particular, that almost his whole Method of Philosophizing was taken from Numbers. And he first of all, as *Laertius* relates, abstracted Geometry from Matter; in which Elevation of the Mind, he found out the 32d, 44th, 47th, and 48th Propositions of the first Book. But he is especially celebrated for the Invention of *Prop.* 32, and 47. of that Book; and he conceiv'd so great Joy

upon

upon this Invention, that, as *Apollodorus* witnesses in *Laertius*, on that Account he sacrific'd an Hecatomb. The same Person first laid open the Theory of incommensurable Magnitudes, and the Five regular Bodies. The same Person did both most diligently teach and exercise the Art of Astrology and Musick: For he did not only acutely and subtily find out many Things himself, but he also first opened a School, in which Youth might learn these honourable and noble Arts.

Pythagoras was follow'd by *Anaxagoras* of *Clazomenæ*, and *Oenopides* of *Chios*, of whom *Plato* makes mention in his Dialogue, *The Lovers*, where young Men are brought in contending about *Anaxagoras* and *Oenopides* in their Descriptions of Circles. *Aristotle* reports, that a certain Treatise of Geometry was written by *Anaxagoras*; and we have it from *Laertius*, that it was shew'd by him that the Sun is greater than *Peloponnesus* (a notable Instance of the Infancy of Astronomy at that Time); and that he made some Conjectures concerning Habitations in the Moon. As for *Oenopides*, to him *Proclus* ascribes the 12 and 23. *l.* 1. These were followed by *Briso*, *Antipho*, and *Hippocrates* of *Chios*, all of them, for attempting the Quadrature of the Circle, reprehended by *Aristotle*, and at the same

time celebrated. But amongst them, *Hippocrates* was by far the most Famous; that celebrated Person, who of a Merchant growing to be a Philosopher and a Geometrician, besides the Quadrature of the Circle, also first attempted the Doubling of the Cube, by two mean Proportionals; which as being an excellent, and indeed the only Way, all that have followed him to this time have embrac'd. 'Tis also his peculiar and great Commendation, that he, as *Proclus* testifies, first wrote Elements, and digested into Order the Discoveries made by others.

Democritus was admirable, not in Philosophy only, but also in the Mathematicks. His Physical Monuments, and, if such there were, his Mathematical Works also, are wholly lost, thro' the Envy (as some report) of *Aristotle*, who desired to have no other Writings read but his own. The Philosophy of *Democritus* hath been restored by *Peter Gassendus*, in a very Learned Work lately publish'd. *Theodorus Cyrenæus*, altho' none of his Mathematical Inventions are extant, yet is great upon this Account, if there were no other, that he is reported to have been the Master of *Plato*.

Unto *Plato* therefore we are come at length, than whom no one brought greater

ter Lustre to the Mathematical Sciences. He enlarg'd Geometry with great and notable Additions, bestowing incredible Study upon it. And above all, the Art Analytick, or of Resolution, was found out by him, the most certain way of Invention and Reasoning. He set off and illustrated his Books of Philosophy in a Mathematical way, and encourag'd whatsoever was admirable in Mathematical Philosophy. Upon the Door of his Academy was read this Inscription: ὐδεὶς ἀγεωμέτρητος εἰσίτω: *Let no one ignorant of Geometry enter here*; an illustrious Instance to demonstrate, how the Mathematicks are not foreign but proper, not unuseful, or unbecoming, but honourable and profitable to sound and certain Philosophy. In a word, how great both Admirer and Master of the Mathematicks *Plato* was, that Man will of himself easily understand, who shall read his Monuments thro'.

Out of *Plato*'s Academy, almost innumerable Mathematicians came forth. Thirteen of *Plato*'s familiar Acquaintance are commemorated by *Proclus*, as Men by whose Studies the Mathematicks were improv'd. From hence were *Leodamus* the *Thasian*, *Archytas* the *Tarentine*, *Theætetus* the *Athenian*, by whom the Mathematicks were notably enlarged. *Leodamus* practised the Analysis received from *Plato*,

and

and is said by *Laertius* to have found out many things by the Help of it. As for *Theætetus*, both his own Inventions, amongst which are the Elements written by him, and the Inscription of regular Bodies; and *Plato*'s Encomiums, who also inscribed a Dialogue to his Name, do make him famous.

Archytas also wrote Elements himself; and his Doubling of the Cube is mentioned by *Eutocius*; whose singular Commendation it likewise was, that he was almost the First that brought down the Mathematicks to humane Uses; by whose Contrivance also a wooden Pigeon was made to fly, as *Gellius* reports; he being followed by *Dædalus*, and other Artificers, yielded Matter for the Fables of the Poets. Moreover, *Archytas* was both a Mathematician and General of an Army: He five times commanded the Forces of his own Citizens, in the Wars of his Country, and five times overcame their Enemies. The meer Name of *Neoclides* is only Famous, he being more illustrious for his Scholar *Leon* perhaps, than for his own Inventions. *Leon* certainly wrote Elements of all the Mathematicks, improv'd them, and made them more fit for Use. Wherefore he is deservedly to be reckon'd amongst the chief Compilers of Elements.

Eudoxus of *Cnidos* was not inferior to *Leon*: A Man great in Arithmetick, and to him (if we may believe the *Greek* Scholiast) we owe the whole fifth Book. He likewise wrote Elements, and made them more general, and increased the Sections begun by *Plato*; over and above this he was the first Framer of Astronomical Hypotheses, and derived down the Springs of Geometry, as *Archytas* had done before, to Mechanicks. *Amyclas* the *Heracleot*, and *Menæchmus*, and his Brother *Dinostratus*, *Helicon* of *Cyzium*, *Theudius*, *Hermotimus* the *Colophonian*, *Philippus* the *Medmæan*, all Platonists, rendered Geometry much more perfect. *Menæchmus* also found out the Conick Sections, and by the help of them, two mean Proportionals; whose Invention in this Case is preferr'd by *Eutocius* before any other. *Theudius* and *Hermotimus* made the Elements more universal and full. And all these, who were of *Plato*'s Academy, brought Mathematick Philosophy to Perfection, as *Proclus* saith. *Xenocrates* also, one of *Plato*'s Auditors, and Master of *Aristotle*, as well as *Aristotle* himself, were famous for the Knowledge of the Mathematicks. When a certain Person, who knew nothing of Geometry, was desirous to be his Auditor, *Go thy way*, saith he, *for thou wantest the very Handles of Philosophy*.

But of *Ariſtotle*, what can I ſay? All his Books are filled with Mathematical Arguments, from a Collection of which *Blancane* hath made a Book. Two of *Ariſtotle*'s School are eſpecially celebrated, *Eudemus* and *Theophraſtus*: This latter wrote two Books of Numbers, four of Geometry, and one of indiviſible Lines: The other, compoſed a Mathematical Hiſtory; and from him *Proclus*, and others have borrowed theirs. To *Ariſteus*, *Iſidore*, *Hypſicles*, moſt ſubtle Geometricians, we are eſpecially indebted for the Books of Solids. Laſtly, *Euclid* gathered together the Inventions of others, diſpoſed them into Order, improv'd them, and demonſtrated them more accurately, and left to us thoſe *Elements*, by which Youth is every where inſtructed in the Mathematicks. He died in the Year before Chriſt 284. There follow'd *Euclid* almoſt an 100 Years afterwards *Eratoſthenes* and *Archimedes*. The Name of *Eratoſthenes* was very famous, but his Writings are loſt. Many Remains we have of *Archimedes*, and many we have loſt.

But when I name *Archimedes*, I conceive in my Mind the very Top of humane Subtilty, and the Perfection of the whole Mathematical Sciences. His wonderful Inventions have been delivered to us by *Polybius*, *Plutarch*, *Tzetzes*, and others. *Conon* was Contemporary to *Archimedes*, one

one who was both a Geometrician and an Aſtronomer, whoſe Death *Archimedes* laments in his Book of the Quadrature of the *Parabola*. There followed *Archimedes* and *Conon*, and that at no great Diſtance, *Apollonius* of *Perga*, another Prince in Geometry, who was called by way of high Encomium, *The Great Geometrician*. There are extant Four [now Seven] moſt ſubtle Books of his Conicks. To the ſame Perſon are aſcribed, the 14 and 15 Books of *Euclid*, which were contracted by *Hypſicles*. *Hipparchus* and *Menelaus* wrote, this latter 6, the other 12 Books of Subtenſes in a Circle; for which Invention, ſo very profitable and neceſſary, great Commendations and Thanks are due to both. There are alſo extant three Books of *Menelaus* concerning Spherical Triangles. Three moſt uſeful Books of Sphericks of *Theodoſius* the *Tripolite* are alſo in the Hands of all. And theſe indeed, if you except *Menelaus*, lived all of them before Chriſt.

In the Year after Chriſt 70, there appeared in the World *Claudius Ptolemæus*, the Prince of Aſtronomers, a Man certainly wonderful, and (as *Pliny* ſaith) above the Nature of Mortals. But he was not only moſt skilful in Aſtronomy, but in Geometry alſo; which as many other Things written by him do witneſs, ſo eſpecially do the Books of Subtenſes: Thoſe of *Mene-*

laus

laus which were Six, and the Twelve of *Hipparchus*, all contracted by him into Five Theorems. As for *Plutarch*, a most fam'd Philosopher, there are extant his Mathematical Problems. And all know of the learned Commentaries of *Eutocius* the *Ascalonite* upon *Archimedes*. By him are recited the Inventions of *Philo*, *Diocles*, *Nicomedes*, *Sporus*, *Heron*, as of so many excellent Masters in the Mathematicks, concerning Doubling the Cube. *Heron*'s Genius certainly was excellent, as well for Mechanicks as Geometry. The Doubling of the Cube delivered by him is commended by *Pappus*, Book III. *Prop.* 7. before all other. The admirable Works of *Ctesibius* the *Alexandrian*, to whom we owe our Pumps, are celebrated by *Vitruvius*, *Proclus*, *Pliny*, and *Athenæus*. The Name also of *Geminus* is not in the lowest Place amongst Mathematicians, whom *Proclus* has preferr'd in many Things before *Euclid* himself.

Diophantus, and he also an *Alexandrian*, was as great in Arithmetick as *Archimedes*, *Apollonius*, or *Euclid* in *Geometry*; he was certainly a Master of all Subtilty relating to Numbers: by him was found out that admirable Art, which we call *Algebra*; which in these Times has been rendered more perfect and universal by *Francis Vieta*, and *Renatus Cartesius*. There are others

others who are celebrated amongst the Antients also; as *Nicomachus*, famous for Arithmetical, Geometrical, and Musical Monuments; *Serenus* well known to Geometricians for his Two Books, concerning the Section of a Cylinder; *Proclus*, *Pappus*, *Theon*. How great a Mathematician *Proclus* was, is manifest from his learned Commentaries on *Euclid*, and other Writings. And this is he, I suppose, who, as *Zonaras* reports, and from him *Ramus*, and *Baronius*, about the Year of Christ 514, with Optic Artifice, and the Glasses which he used, burnt the Fleet of *Vitalian*, who was besieging *Constantinople*. The Praises of *Theon*, which truly are deservedly great, *Peter Ramus* wonderfully exaggerates; insomuch that even the Books which hitherto all have ascribed to *Euclid*, ought, as he thinks, to be attributed to *Theon*. But *Ramus*, who every where is ready to detract from *Euclid*, and this without grounding himself upon any solid Foundation, is not to be hearken'd to here. To come at length to a Conclusion, let *Pappus* bring up the Rear, the last in Time among the Antients, as being one who liv'd about the Year 400; but in Reputation, and all Mathematical Commendation, to be reckon'd amongst the first. *Alexandria*, that City so fruitful of great Men, which before had brought forth *Hypsicles*, *Ctesibius*

and

and *Diophantus*, produced him also, to the great Advantage of the Mathematicks. He wrote Seven Books of Mathematical Collections, of which the Two First are lost. The Five other do abound with so many, and such various most noble Inventions in almost all Parts of the Mathematicks, that they are esteemed amongst the chief Monuments of the Antients which are extant.

And thus you have a short History of the Origin and Progress of the Mathematicks. From which appears the Antiquity, Excellency, and Dignity of this Science. And truly the same eminent Persons in the Commonwealth of Learning, who discover'd Philosophy, discover'd also the Mathematicks, like two Sisters born at one Birth; whom if any one would violently separate from each other, he certainly attempts to break off their native Concord, with most notable Injury, and as it were Cruelty to both; seeing, as it is wont to fall out in the Case of Twins, where they are remov'd from one another, in Place or by Death, so it will be like to happen here, that Mathematicks being plucked away from her, Philosophy must needs languish and pine away.

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Dr. BARROW's Words, prefix'd before his *Apollonius.*

God always acts Geometrically.

HOW *great a Geometrician art thou, O Lord! For while this Science has no Bounds; while there is for ever room for the Discovery of New Theorems, even by Human Faculties; Thou art acquainted with them all at one View, without any Chain of Consequences, without any Fatigue of Demonstrations. In other Arts and Sciences our Understanding is able to do almost nothing; and, like the Imagination of Brutes, seems only to dream of some uncertain Propositions: Whence it is that in so many Men are almost so many Minds. But in these Geometrical Theorems all Men are agreed: In these the Human Faculties appear to have some real Abilities, and those Great, Wonderful and Amazing. For those Faculties which seem of almost no force in other Matters, in this Science appear to be Efficacious, Powerful, and Successful,* &c. *Thee therefore do I take hence occasion to Love, Rejoice in, and Admire; and to long for that Day, with the Earnest Breathings of my Soul, when thou shalt be pleased, out of thy Bounty, out of thy Immense and Sacred Benignity, to allow me to behold, and that*

with a pure Mind, and clear Sight, not only these Truths, but those also which are more numerous, and more important; and all this without that continual and painful Application of the Imagination, which we discover these withal, &c.

Mathematical Notes or Abbreviations.

$=$ The Note for Equality. So $a = b$ signifies that a and b are equal.

$+$ The Note for Addition. So $a + b$ signifies the Sum of a and b together.

— The note for Subtraction. So $a - b$ signifies the Difference between a and b.

× The Note for Multiplication. So $a \times b$ or ab signifies a multiplied by b.

:: The Note for equality of Proportion. So $A : B :: a : b$ signifies that A bears the same Proportion to B, that a bears to b.

$\div\!\div$ The Note of continued Proportion. So $A, B, C \div\!\div$ signifies that A bears the same Proportion to B, that B bears to C.

q The note for a Square. So CBq signifies the Square of the Line CB.

c The Note for a Cube. So CBc signifies the Cube of the Line CB.

THE

The Elements of Euclid.

BOOK I.

Definitions.

1. A Point is a Mark in Magnitude, which is [supposed to be] indivisible.

That is, which cannot be divided so much as in Thought. A Point is the beginning, as it were, of all Magnitude, as Unity is of Number.

2. A Line is a Magnitude which hath Length only, and wants all Breadth; forasmuch as it is understood to be produced from the flowing of a Point.

3. Points are the Terms of a Line.

4. A right Line, is that which lies evenly betwixt its Terms. Fig. 1. Table 1.

Or as *Archimedes*: A right Line is the least of all those which have the same Terms; or, is the shortest of all those which can be drawn betwixt two Points.

Or as *Plato* hath it: A right Line is that whose Extremes hide all the rest; [that is, when the Eye is placed in a Continuation of the Line.]

The Sense is the same in all. The Instrument whereby right Lines are described, is [called] a Rule; which whether it be strait or not you may know by this Tryal.

Describe a Line according to the Rule; then turning the Rule so, that that which before was the Right-hand End may now become the Left-hand End, apply it again to the Line before described; if it doth now entirely

fall in with the Line, the Rule is ſtrait; if not, the Rule is not ſtrait. The Reaſon hereof depends on Axiom 13.

5. A Surface is a Magnitude which hath only Length and Breadth.

It hath two Dimenſions therefore: And is underſtood to be produc'd by the flowing of a Line.

6. Lines are the Extremes of a Surface.

7. A Plane, or a plain Surface, is that which lies evenly betwixt its extreme Lines.

Or as *Hero*, that, to all the Parts whereof a right Line may be accommodated.

For it is produc'd from the Motion of a right Line.

Or, A plain Surface is that whoſe Extremes any of them hide all the reſt, [the Eye being placed in a Continuation of the Surface.]

Or, It is the leaſt of all Surfaces which have the ſame Terms. The Senſe is the ſame in all.

Euclid hath not here defined a Body or Solid, becauſe he was not yet about to treat concerning it. But leſt any one ſhould want the Definition thereof, take it here thus: A Body is a Magnitude long, broad and deep. A Body therefore hath three Dimenſions, a Surface two, a Line one, a Point none.

8. A plain Angle is the mutual Inclination to each other of two Lines, which touch one another in a Plain; but ſo as not to make one Line.

Fig. 2, 4. Therefore the two Lines AB, CA, touching one another in A, but ſo as not to make one Line, conſtitute an Angle.

9. The Sides or Legs of an Angle are the Lines which make the Angle.

10. The *Vertex* or Top of an Angle is the Point (A) in which the Legs do meet and touch one another.

Note, that a ſingle Angle is deſigned by one Letter put at the Top: When there are more at one Point, they are deſigned by three Letters, the middlemoſt of which denotes the Top of the Angle; and many times alſo by one Letter interpos'd betwixt the Sides near the Top. So in Fig. 5. the Angle made by the Lines BA, CA, is deſigned either by three Letters BAC, or by one only O.

11. Angles are called Equal, if when the Tops of them are laid upon one another, the Sides of one agree with the Sides of the other. But unto this it is not required that the Sides ſhould be of an equal Length.

3 12. They

12. They are called Unequal when the Top and one Side agreeing, the other doth not agree; and that is called the Greater, whose Side falls without. So the Angle B A E is greater than the Angle B A C. *Fig.* 5.

An Angle is not diminish'd or increas'd by the Diminution or Augmentation of the Sides that include it.

13. A right-lin'd Angle is that which right Lines constitute; a curvi-linear, which crooked Lines; a mixt one, that which a right Line and a crooked one make. *Fig.* 2, 4.

14. When the right Line [C A] standing upon the Right one [B F] leans unto neither Part, and therefore makes the Angles on both Sides equal, C A B = C A F, both of the equal Angles are called Right ones: But the right Line C A which stands upon the other, is called a perpendicular Line, or barely a Perpendicular. *Fig.* 6.

A right Angle may also be defined thus. *Fig.* 6.

A right Angle is that (B A C) to which on the other Side an equal one ariseth (C A F) if you produce or draw forth a Side, as (B A).

Two Rules so joined as to contain a right Angle, make an Instrument, which is called a Square. *Pythagoras* was the Inventor of it, as *Vitruvius* affirmeth, *c.* 2. *l.* 9. So great is the Use and Force of a right Angle in Framing, Measuring, and Strengthning all Things, that nothing almost can be done without it. The Proof of a Square is made thus: Apply the Side of it, A E to the right Line A F, and describe the right Line C A along the other Side. Then turning the Square towards B, if on both Sides it agrees to the right Lines C A, A B, you may know that it is true and exact. The Reason hereof appears from the 14th Definition it self.

15. The Angle B A C, which is greater than the right one F A C, is called an obtuse Angle. *Fig.* 7.

16. The Angle (L A C) which is less than the right Angle (F A C) is called an Acute one. *Fig.* 8.

17. A plain Figure is a plain Surface, bounded on every Side with one or more Lines.

18. A Circle is a plain Surface contained within the Compass of one Line called the Circumference; from which Line all the right Lines that can be drawn unto one certain Point, within the contained Space (A), are equal. *Fig.* 9.

19. That Point is called the Center.

Fig. 9. 20. The Diameter is a right Line (B A) drawn thro' the Center, and on both Sides terminated at the Circumference; and consequently it divides the Circle into two equal Parts, (as is abundantly manifest from the exact Agreement of two Semicircles when laid one upon another.)

21. The Semi-diameter or *Radius* is the right Line A F drawn from the Center to the Circumference.

22. A Semi circle is a Figure (B L C) which is contain'd by the Diameter B C, and half the Circumference (B L C.)

Mathematicians are wont to divide the Circumference into 360 equal Parts (which they call Degrees) the Semi circumference into 180, the Quadrant or Quarter into 90.

23. A Right-lin'd Figure is a plain Surface bounded on every Side with right Lines.

Fig. 10. 24. A Triangle is a plain Surface contained by Three right Lines,

This is the first and most simple of all Right-lin'd Figures, and that into which they are all resolv'd.

Fig. 10. 25. An Equilateral Triangle is that which hath all the Sides equal.

Fig. 11, 12. 26. An *Isosceles* or equicrural Triangle is that which hath only two Sides equal.

Fig. 13. 27. A *Scalenum* is that which hath Three unequal Sides.

Fig. 13. 28. A right-angled Triangle is that which hath one Angle right.

Fig. 12. 29. An obtuse-angled Triangle is that which hath one obtuse Angle.

Fig. 10, 11. 30. An acute-angled Triangle is that which hath three acute Angles.

Fig. 14, 15. 31. Amongst quadrilateral Figures, the Rectangle is that which hath Four right, and consequently equal Angles; whether the Sides be equal or not.

Fig. 15. 32. A Square is that which hath equal Sides, and is Right-angled, and consequently Equi-angled.

Every Square is a Rectangle; but every Rectangle is not a Square.

Fig. 16. 33. A Rhombus is a quadrilateral or four-sided Figure, which is equilateral, but not equiangled.

Fig. 17. 34. A Rhomboides is that which hath the opposite Sides and Angles equal, but is neither Equilateral, nor Equiangled.

35. A

35. A Parallelogram is a quadrilateral Figure, which hath each Two of its opposite Sides (A B, F C, and B F, A C) parallel to each other. Now what parallel Lines are, will be shewed in the following Definition. Fig. 14, 15, 16, 17.

Every Rectangle and Square is a Parallelogram; but every Parallelogram is not a Rectangle or a Square.

36. Right Lines are Parallel or Equi-distant, which being in the same Plane, and drawn out on both Sides infinitely, are distant from one another by equal Intervals. Fig. 18.

The Intervals are said to be equal, in respect of the Perpendiculars. Wherefore if all the Perpendiculars (Q L) unto one of the two Parallels (A B) shall be equal, the right Lines (A B, C F) are said to be Parallel.

Parallels are produc'd, if the right Line (L Q) which is perpendicular to the right Line (A B) be moved along (A B) always perpendicularly; for then its Extremity L describes the Parallel C F.

37. The Diameter or Diagonal of a Parallelogram, and every Quadrilateral, is a right Line (A F) drawn thro' the opposite Angles. Fig. 17.

38. Plain Figures contain'd by more Sides than Four, are called Many-sided or Many-angled, and by a *Greek* Word *Polygones*.

39. The external Angle of a right-lin'd Figure, is that which ariseth without the Figure when the Side is produc'd. Such are F B C, G C A, H A B. Every Figure therefore hath so many external Angles as it hath Sides, and internal Angles. Fig. 19.

Postulates.

A Postulate is that which is manifest in it self, that it may easily be done, or conceiv'd to be done. It is required therefore to be granted that we may,

1. From any Point given draw a right Line unto any other Point given.

2. Draw forth a finite right Line in Length still farther.

3. From any Center at any Interval describe a Circle.

Axioms.

AN Axiom is a Truth manifest of it self.

1. Those things which are equal to the same thing, are equal also amongst themselves. And that which is greater or lesser than one of the Equals, is also greater or less than the other of them.

2. If to Equals you add Equals, the Wholes will be equal.

3. If from Equals you take away Equals, the Remainders will be equal.

4. If to Unequals you add Equals, the Wholes will be unequal.

5. If from Unequals you take away Equals, the Remainders will be unequal.

6. What things are each of them half of the same Quantity, are equal amongst themselves; and what things are double, or treble, or quadruple of the same, are equal amongst themselves.

7. What things do mutually agree with one another, are equal.

8. If right Lines be equal, they will mutually agree with one another; and the same thing is true of Angles.

9. The whole is greater than its part.

10. All right Angles are equal amongst themselves.

11. Parallel Lines have a common Perpendicular: That is, the right Line which is perpendicular to one of them, is perpendicular also to the other.

Fig. 21. 12. The two perpendicular Lines (LO, QI) intercept equal Parts of the Parallels.

13. Two right Lines do not comprehend a Space; for unto this there are required three at the least.

14. Two right Lines cannot have one common Segment; for that they cut one another only in a Point.

Of Propositions some propose something to be done, and are called Problems; in others we proceed no further than bare Contemplation, which therefore are named Theorems.

PRO-

PROPOSITIONS.

THE requisite Citations are found in the Margin: When Propositions are cited, the first Number designs the Proposition; the Letter l. with the Number following, signifies the Book. As when you meet with (*per* 5. *l.* 3.) you must read it thus, (by the 5th Proposition of the 3d Book.) The Figure is always to be sought amongst the Figures of that Book in which we are then conversant. The rest of the Citations are easy to be understood.

The primary Affections of Triangles and Parallelograms are deliver'd in this Book. The more famous Propositions are, 32, 35, 37, 41, 44, 45, 47.

PROPOSITION I. Problem. *Fig.* 23.

Upon a given Right Line (*A B*) *to make an Equilateral Triangle.*

From the Centre A, with the Interval (A B) (*a*) describe the Circle F C B: and from the Centre B with the same Interval B A describe the Circle A C L, cutting the former in the Point C, from which Point draw the right Lines C A, C B. (a) *Per Postul.* 3.

I say, that the Triangle A C B now made, is Equilateral. For the right Line A C is equal to the right (*b*) Line A B, seeing they are Semi-diameters of the same Circle F C B. And again, the right Line B C is equal to the same right Line B A, seeing they are both Semidiameters of the Circle L C A. Therefore A C, B C are (*c*) equal betwixt themselves. And therefore all the Sides of the Triangle are equal. Therefore the Triangle (*d*) A C B is both Equilateral, and made upon the given Line A B; which was the thing to be done. *Q.E.F.* (b) *Per Def.* 18. (c) *Per Axiom* 1. (d) *Per Def.* 25.

Corollary. *Hence we may measure an inaccessible Line, as A B. For suppose any Equilateral Triangle whatsoever B D E applied to the Point B along the Line B A. Looking from the Point B along the Line B E, mark as many Points as you conveniently can in* *Fig.* 77.

the Line B C. Then remove the Triangle B D E along the Line B C, from one place to another of that Line, until by taking aim along the side of the Triangle E D or C F, you see the inaccessible Point A in a Continuation of that Line. Thus the Triangle B A C is as well Equilateral as B D E. If therefore you shall now measure the accessible Line B C, you have the Measure of the inaccessible A B. Q. E. F.

PROP. II. Problem.

Fig. 24. *FRom a given Point* A *to draw a right Line equal to one given E F.*

Take with a Pair of Compasses the Interval E F, and transfer it from A to D, the right Line A D will be equal to the given E F.

PROP. III. Problem.

Fig. 24. *TWO unequal right Lines being given, from the greater of them G H to cut off G I equal to the less E F.*

Take with a Pair of Compasses the Interval of the lesser given Line E F, and transfer it unto the greater from G to I.

PROP. IV. Theorem.

Fig. 25. *IF in two Triangles (X, Z) one side of the one (B A) be equal to one side F L of the other, and another side (C A) of the one equal to another side (I L) of the other, and the Angles (A and L) made by those sides be also equal; then the Bases (B C, F I) are likewise equal, as also the Angles at the Bases (B, F, and C, I) which are opposite to equal sides, and consequently the whole Triangles are equal.*

For if we suppose the Triangle Z to be laid upon the Triangle X, the Sides L F, L I will perfectly agree and fall

fall in together with the Sides that are equal to them, A B, A C, and this in such sort (c) that the three Points (L, F, I) shall fall upon the three Points, (A, B, C). Therefore the whole Base F I will also fall upon the whole Base B C. But then the Angles F, B, and likewise those I, C, and the whole Triangles will mutually (*congruere*) agree to each other. All therefore by Axiom 7th are equal. *Q. E. D.* Which was the Thing to be demonstrated. (c) Per Axio. 8.

Coroll. (1.) *Hence we may also in another way measure the Line A B, altho otherwise impracticable by reason of some Obstacle, as a River,* &c. *between the Extremities thereof. For from any Point whatsoever, as the Point C, let the Angle A C B be observed, and then let the Lines AC, B C be measured: and in any accessible Plane let there be measured about the Angle F, which is equal to the Angle C, two Lines F D and F E, which are equal to the Lines AC and B C respectively. And then there will be the accessible Line D E equal to the inaccessible A B.* Q. E. I. Fig. 78.

Coroll. (2.) *Hence also, those who play at Billiards with Ivory Balls may learn how by the Reflexion of their own to hit and remove their Adversaries Ball. For let B be the Ball to be striken, A that which is to strike it, and C D the Rectilinear Plain. Let the Line B E be perpendicular to the Line C D, and D E be equal to D B. If the Ball A be stricken and carried along the right Side A F E unto the Point F, it will there be so reflected that after the Reflexion it will tend unto B. For in the Triangles B F D, E F D, the Side F D is common to both, and the Side D B is equal to the Side D E; and the Angles at D are equal, as being right ones. The whole Triangles therefore are equal: and therefore the Angle B F D, which is equal to the Angle D F E, is * equal to A F C, the Angle A F C being vertically opposite to D F E. Wherefore, seeing the Angle A F C is the Angle of Incidence, which in such cases is equal to the Angle of Reflexion, it is manifest that B F D, which hath been proved equal to A F C, is the Angle of the Reflexion of the Ball A, and that the Ball tending towards E is in the Point F so reflected as to hit the Ball B.* Q. E. D. Fig. 79. * Per 15. l. 1.

Scholium

Scholium or Observation.

BY much the same way of Reasoning whereby this 4th Proposition has been demonstrated, the following Theorem, which we shall have occasion to use by and by, may be demonstrated also.

Fig. 25. If in Two Triangles X, Z, the Sides BC and FI shall be equal, and the Angles adjacent to these Two Sides equal also, *viz.* B and C equal to F and I; all the other Things, and the whole Triangles themselves will be equal.

For the Side FI laid upon the Side BC will agree, or thorowly concide with it (*a*). And then because the Angles B and C are equal to those F and I, when the Side FI is laid upon the Side BC: FL (*b*) will fall exactly upon BA, and IL upon CA. Therefore the Point L will fall upon the Point A (for if it fall without A, the Sides FL, IL would not fall upon the Sides BA, CA). Therefore all Things are equal by Axiom 7th.

(a) Per Axi. 8.

(b) Per Axi. 8.

PROP. V. Theorem.

Fig. 26. *IN an Isosceles or Equicrural Triangle, the Angles at the Base (A, C) are equal.*

Let the Triangle ABC be understood to be twice put, but in an inverted Posture *cba*. Because therefore in the Two Triangles ABC, *cba*, the Side AB is by the Supposition equal to the Side *cb*, and the Side CB to the Side *ab*, and the Angle B to the Angle *b*; the Angle A also at the Base will (*c*) be equal to the Angle *c*. Q. E. D. For as for the Angles C and *c*, they are the same.

(c) Per 4. l. 1.

Corollary.

THEREFORE an Equilateral Triangle is also Equiangular.

PROP. VI. Theorem.

Fig. 26. *IF in a Triangle (ABC) Two Angles (A and C) be equal, the Sides (AB, BC) which are opposite to those Angles are equal also.*

Let

Let the Triangle A B C be supposed to be twice put, but in an inverse situation, *c b a*; because therefore in the Triangles A B C, *c b a*, one Side A C is equal to one Side (*c a*) and the Angle A is equal to the Angle *c*, and the Angle C equal to the Angle *a*, all the other Things shall be likewise (*a*) equal, and consequently A B shall be equal to the Side *c b*. *Q. E. D.* For as for the Lines C B and *c b* they are the same. (a) Per Schol. Prop. 4.

Coroll.

THEREFORE an Equiangled Triangle, is also Equilateral.

Coroll. (2.) *Hence, by the means of the Shadow of the Sun, we may measure the Height of a Tower, or any elevated Point. For when the Sun is elevated 45 Degrees above the Horizon, the Shadow which the Tower casts towards the Horizon will be exactly equal to its Height. For, by reason that the Angle ACB is half a right Angle, the Angle BAC also * will be half a right one; and so, by the force of the present Proposition, the Line AB will be equal to the Line BC. The Line BC therefore being found by measuring, there is found at the same time the Line AB, the Height of the Tower above the Horizon.* Fig. 80. * Per Coroll. 11. Prop. 32. l. 1.

Coroll. (3.) *The same Thing also may be found without the Sun by the means of an Astronomical Quadrant. For where the Angle of Elevation is half-right, there the Height of the Tower above the Observer's Eye is equal to the distance of the same Eye, from that Part of the Tower which is opposite to it. The Distance therefore of the Eye from the Tower being given by measuring, there is given at the same time the Height of the Tower.* Q. E. I.

The VIIth Proposition in *Euclid* is for the sake of the VIIIth, which without it will here be demonstrated.

PROP. VIII. Theorem.

IF Two Triangles (X, Z) have all their Sides equal amongst themselves respectively (AC equal to EF; CB to FI; AB to EI;) they will also have all the Angles which are opposite to equal Sides, equal: (C equal to F; A to E; B to I.) Fig. 27.

For

For suppose the Side A B laid upon its Equal E I, if then the Point C falls upon F, the Triangles will in the Whole agree or coincide, and consequently all the Angles will be equal. But the Point C will fall upon the Point F. For,

Fig. 81. *From the Centre A let a Circle be described with the Semidiameter E F; and from the Centre I let another Circle be described with the Semidiameter IF; the Point C by reason of the Equality of the Sides of both Triangles, will be in the Circumference of both Circles, and consequently in the Point E, the common Intersection of both these Circumferences.* Q. E. D.

PROP. IX. Problem.

Fig. 29. TO *Bisect or Divide into two equal Parts a given right-lin'd Angle, as I A L.*

From the Sides of the Angle take with a Pair of Compasses two equal Lines, A B, A C; then from the Centres B and C describe two equal Circles cutting one another in F; which done draw the Line F A. This bisects the Angle.

For draw the Lines B F, C F; the Triangles F A B, F A C are to each other Equilateral; for the Sides A B, A C are by the Construction equal, as in like manner are the Sides B F, C F, they being Semidiameters of equal Circles; and AF is common to both Triangles. Therefore the Angles B A F, C A F (d) are equal. Therefore the given Angle I A L is bisected. *Q. E. F.*

(d) Per 8. l. 1.

Corollary.

HENCE we learn how an Angle may be divided into 4, 8, 16, &c. equal Angles, *viz.* by bisecting each Part again.

Scholium.

NO one hath hitherto taught the way of dividing Angles into all equal Parts whatsoever with a Pair of Compasses, and a Rule.

Fig. 30. Yet may you divide any given Angle mechanically into any equal Parts whatsoever, if from the Top of the Angle as the

the Centre you describe an Arch between the Legs of the Angle, and divide that Arch into as many equal Parts as you require; for right Lines let down from A thro' the Points of the Division, will cut the Angle into so many equal Parts.

PROP. X. Problem.

To bisect a finite given Line (A B.) Fig. 31.

Upon the given A B make an Equilateral (*a*) Triangle A G B. Bisect its Angle G (*b*) with the right Line G C. The same shall bisect the given Line A B. (a) Per 1. l. 1. (b) Per praeced.

For in the Triangles X, Z, the Side C G is common; and by the Construction G B, G A are equal, and the Angles contained between them A G C, B G C, are likewise equal. Therefore the Bases A C, B C (*c*) are equal. The given Line therefore A B is bisected. Q. E. F. (c) Per 4. l. 1.

But for Practice it is sufficient from the Centers A and B to describe two equal Circles, cutting one another in G and L, and so to draw the right Line G L.

PROP. XI. Problem.

From a given Point (A) in a given right Line (L I) to raise a Perpendicular. Fig. 32.

With a Pair of Compasses take the equal Lines A C, A F. From the Centre C and F describe two Circles, cutting one another in B. The Line which is drawn from B to A will be the Perpendicular required.

For let the right Lines C B, F B be drawn. The Triangles X and Z are equilateral to one another. Therefore the Angles C A B, F A B are equal (*a*.) Therefore B A is (*b*) perpendicular to the Line (L I.) *Q. E. F.* (a) Per 8. l. 1. (b) Per def. 14.

In Practice this and the next are easily performed by the help of a Square.

PROP.

PROP. XII. Problem.

Fig. 33. *FROM a given Point (A) which is without an infinite right Line (as L Q) to let fall a Perpendicular to that Line.*

From the Centre A describe a Circle which may cut the given L Q in C and I. Bisect the right Line C I (c) with the right Line A B. This A B is the Perpendicular required.

(c) Per 10. l. 1.

For let there be drawn A C, A I. Because by the Construction X and Z are equilateral to one another; Therefore the Angles (*d*) C B A, I B A, are equal. Therefore, A B is (*e*) Perpendicular, *Q. E. F.*

(d) Per 8. l. 1. (e) Per Def. 14.

PROP. XIII. Theorem.

Fig. 34. *THE right Line (B A) standing upon the right Line (C F) either makes two right Angles, or Angles equal to two right ones.*

For if B A stand upon it perpendicularly, then by Definition 14 the two Angles B A C, B A F will be right ones. And if B A stand obliquely, let there be rais'd (*f*) the Perpendicular A L. Where because the unequal Angles C A B, F A B possess the same Place which the two right ones C A L, L A F do, and agree to them, they are equal (*g*) to them. *Q. E. D.*

(f) Per 11. l. 1. (g) Per Axi. 7.

Corollaries.

1. IN the same manner it will be demonstrated, if more right Lines than one stand upon the same right Line, that the Angles thereby made are equal to two right ones.

Fig. 37. 2. Two right Lines cutting one another, make the Angles equal to four right ones.

Fig. 36. 3. All the Angles which are about one Point, make Angles equal to four right ones. It appears from Corollary 2.

4. The

4. The Angle C A F being known, you at the same time know its Compliment unto two right Angles B A F. For Example, Let the Angle C A F be of 70 Degrees; the Angle B A F will be of 110 Degrees. For those two Numbers added together make 180 Degrees, which is the Measure of two right Angles. Fig. 37.

PROP. XIV. Theorem.

If two right Lines (XR, ZR) at the same Point of a right Line QR make the Angles on both Sides (XRQ, ZRQ) equal to two right Angles; the Lines (XR, ZR) make one right Line. Fig. 35.

If you deny it, let X R, B R make one right Line. Therefore the Angles X R Q, Q R B (*a*) will make two right Angles. Which thing is (*b*) absurd; seeing by the Hypothesis X R Q, Z R Q do make two right Angles.

(a) *Per* 13. 1.
(b) *Contra Axio.* 9.

PROP. XV. Theorem.

If two right Lines (BC, FL) cut one another in A, the Angles opposite at the top (A) are equal, viz. *LAB to CAF, and BAF to LAC.* Fig. 37.

For because B A stands upon the right Line L F, the Angles L A B, F A B are (*c*) equal to two right ones: And because F A stands upon the right Line B C, the Angles F A C, F A B are also equal (*d*) to two right ones. Therefore the two Angles together (*e*) L A B, F A B are equal to those two together C A F, F A B. Taking away therefore the common Angle F A B, there remains (*f*) L A B equal to C A F. In the same manner B A F, L A C are shewed to be equal.

(c) *Per* 13. 1.
(d) By the same Prop.
(e) *Per axi.* 1.
(f) *Per axi.* 3.

Coroll. *From these two Propositions we gather in Catoptricks, that a Ray of Light, as reflected in an Angle equal to the Angle of Incidence, taketh the shortest way of all. e. g. When the Angles BEC, AEF are equal, the Lines AE and EB taken together, are shorter than any Lines whatsoever, as AF and FB taken together. For from the Point B let the perpendicular Line* Fig. 82.

Line B C be let down; and let B D and D C be equal: Let the Lines also E C and F C be drawn. Now in the Triangles B E D and D E C, seeing the Side D E is common to both, and the Side B D and D C are equal by the Hypothesis, as is also in the like manner B D E equal
* Per 4. l. 1. *to the Angle C D E; the Triangles shall be * equal in all other things, and B E shall be equal to C E, and the Angle B E D to the Angle D E C: (where because the Angle D E C is equal to [B E D, that is] A E F, the Lines A E, E C are prov'd to make one right Line.) And in the same manner the Line B F will be proved equal to F C. Seeing therefore the Lines B E and E A taken together, are equal to the Line C A, and the Lines B F, F A taken together are equal to the Lines C F, F A taken together; It is manifest that C A, which*
† Per 20. l. 1. *is one Side of the Triangle A C F †, is less than the two Sides C F, F A taken together.* Q. E. D.

PROP. XVI, XVII.

T*HESE two* Propositions *are contain'd in* Proposition 32; *and are not here made use of till then.*

PROP. XVIII. Theorem.

Fig. 38. I*N every Triangle the Angle (A) which is opposed to the greater Side (B O) is the greater; and that (B) which is opposite to the lesser Side (A O) is the lesser Angle.*

(*A*) Cannot be equal to (B) for then the opposite
(a) Per 6. l. 1. Sides B O, A O would be equal *(a)*; which is contrary to the Hypothesis. Neither can A be less than B, for if it were so, there might within the Angle B be made an Angle A B F by the right Line B F; which Angle should be equal to A. But then by the 6th of this Book B F, A F shall be equal; and if you add to both O F, then B F, F O shall be equal to A O. But A O by the Hypothesis is less than B O. Therefore B F, F O shall be less than B O, which contradicts the Definition of a

right Line, which is the shortest of all betwixt two Points. Therefore the Angle A is neither less than B, nor equal to it. Therefore it is greater. *Q. E. D.*

PROP. XIX. Theorem.

IN the Triangle A O B the Side (B O) which is opposed to the greater Angle (A) is the greater; And that (A O) which is opposed to the lesser Angle B, is the lesser. Fig. 38.

This Proposition is the Converse of the former. B O is not less than A O, for if it were, the Angle (A) by the 18th would be less than B; which is contrary to the Hypothesis. Nor can B O be equal to A O, for in this Case by the 5th, the Angles A and B would be equal. But this Equality of those Angles is contrary to the Hypothesis. Therefore B O is greater than A O. *Q. E. D.*

Coroll. *Hence we gather that a Globe or Ball perfectly polished cannot rest in an horizontal Plane perfectly polished, but where it touches the Earth. For let the Line A B be an horizontal Plane, C the Earth's Centre, C A the Semidiameter of the Earth, perpendicular to the Tangent A B. The Globe placed at B, because of its Gravity, and the Declivity of the Plane, will descend towards A. For in the Triangle C A B the perpendicular Line C A, which is opposite to the acute Angle A B C, is less than the Line B C which is opposed to the right Angle B A C; and so there is from B to A a perpetual Descent, in which the Globe cannot rest. And in the like manner we prove the Descent of Fluids, and their Conformation into a spherical Surface.* Fig. 83.

PROP. XX. Theorem.

IN any Triangle, any two Sides of it taken together, are greater than the remaining Side.

This with *Archimedes* is as it were an Axiom; forasmuch as it is immediately manifest out of his Definition of a right Line; which see above amongst the Definitions.

PROP. XXI. Theorem.

Fig. 39. *IF from the Ends of one Side AB, two right Lines be drawn, and joined together within the Triangle (as the Lines AO, BO); these are less than the Sides of the Triangle (AC, BC), but they comprehend a greater Angle (AOB).*

For as for the first Part of the Proposition, Draw out AO unto F: AC, CF are (*a*) greater than AF. Therefore the common Line FB being added, AC, BC are greater than AF, FB. Again, OF, BF are greater (*b*) than OB. Therefore the common AO being added, AF, BF are greater than AO, BO. Therefore AC, CB are much greater than AO, OB.

(a) Per 20. l. 1.

(b) By the same.

The second Part of this Proposition will be demonstrated in the second Corollary of the first Part of Proposition 32. And in the mean while we shall make no use of it.

PROP. XXII. Problem.

Fig. 40. *TO make a Triangle of three given right Lines (BO, LB, LO) (of which any two must be greater than the third.)*

Let BL one of the given Lines be taken, and B one of its Extremities being taken for the Centre, with the Interval of the other given Line BO describe an Arch. Then the other Extremity L being taken for the Centre, with the Interval of the third given Line LO describe an Arch, cutting the former in O; which being done, and the right Lines BO, LO being drawn, I say that that is done which was to be done.

The Demonstration is manifest from the Construction.

PROP.

PROP XXIII. Problem.

AT a given Point in a right Line (as B) to make an Angle equal to a given one (A). Fig. 40.

First of all let C F be drawn at a venture, cutting the Sides of the given Angle A. Then in the given right Line from B take B L equal to A F. Then from the Centre B describe a Circle with the Interval A C; afterwards another from the Centre L with the Interval F C, which may cut the former in O. Then from O unto B and L having drawn right Lines, the Angle L B O will be equal to the given one A.

For by the Construction the Triangles are Equilateral to one another. Therefore by the 8th of this Book the Angles B and A are equal.

Scholium.

IT seems meet for the Sake of beginners to propound some things here which are necessary for Practice about Angles.

The Measure of an Angle is the Arch of a Circle, Fig. 41. which is described from A, the Top of the Angle as the Centre. Therefore look how many Degrees the Arch B C which is intercepted between the Legs of the Angle B A C shall contain, of so many Degrees the Angle B A C shall be said to be. And so because B F a quarter of the Circumference, contains 90 Degrees, and measures the right Angle B A F, a right Angle shall be said to be of 90 Degrees. In like manner, because half the Circumference, which is divided into 180 Degrees, measures two right Angles, and the whole Circumference, which is divided into 360 Degrees, measures four right Angles; two right Angles shall be said to make 180 Degrees, and four, 360 Degrees. These things being premised, the Practice about Angles is as follows.

1. At *B* a given Point in a right Line to make an Fig. 42. Angle equal to the given one A.

From A the Top of the given Angle as the Centre describe betwixt the Sides the Arch C F. Then from B

the given Point as the Centre describe with the same Interval the Arch L Z; from which take off L O equal to C F. Thro' B and O draw a right Line; L B O shall be equal to the given A.

Fig. 43. 2. To examine the Degrees of the given Angle O P Q. This is done very easily by any Semicircle or Protractor, which is divided into 180 Degrees. For put the Centre of the Semicircle upon P the Top of the Angle, and the Radius of the Semicircle P L upon the Side of the Angle P Q; and the Arch L O which is intercepted betwixt the Legs of the Angle will shew of how many Degrees the given Angle is.

3. To frame an Angle containing a given Number of Degrees, as 42.

Fig. 43. Draw the right Line X Q, in which mark the Point P. Upon P put the Centre of the Semicircle, and its Semidiameter P L upon P Q. From L number 42 Degrees, that is, until you come to O. A right Line drawn from P thro' O, will give the Angle O P L of 42 Degrees.

PROP. XXIV, and XXV. Theorems.

Fig. 44. *If two Triangles (B A C, B A F) shall have two Sides (B A, A C) equal to two (B A, A F) each to each; and if one of the Triangles hath the Angle (B A F) contained by those Sides greater than the other (B A C); it shall have the Base B F greater than the Base (B C.)*

And again, If it hath the Base greater, it shall have the Angle greater.

From the Centre A describe a Circle which passeth thro' C, it shall pass also thro' F, because A C, A F are supposed to be equal. Therefore B F shall fall betwixt the Points A and C. Then join C F. The Angle B C F; is greater than the Angle A C F; that is, by the 5th of this Book, than the Angle A F C, and consequently much greater than the Angle B F C. Therefore in the Triangle B C F, (*a*) B F which is opposite to the

(a) Per 19. l. 1.

the greater Angle B C F is greater than B C whch is opposite to the lesser Angle B F C.

2. As for the Second Part of the Proposition this is manifest from the first Part.

PROP. XXVI. Theorem.

If two Triangles (X and Z) have two Angles equal to two, one Angle of the one equal to one Angle of the other (B to F and C to I), and one Side of one Equal to one of the other, whether it be that which is betwixt the equal Angles (as BC=FI) or a Side which is opposed to one of the equal Angles (as AC=LI); all the other Parts shall be equal. Fig. 25.

For first, let the Sides (BC, FI) which are betwixt the equal Angles be supposed equal: In this Case all the other Parts are equal; as hath been already demonstrated in the *Scholium* of Proposition 4.

Again, suppose the Sides A C, L I which are opposed to the equal Angles to be equal. Here because the Angles (B, C) are by the Hypothesis equal to (F, I) the other Angles, also (A, L) shall be equal by Coroll. 9. Prop. 32. which Proposition depends not upon this. Therefore by the first Part of this all the other Parts are equal.

Coroll. *Hence also, following* Thales, *we may measure inaccessible Distances. e g. Let AD be an inaccessible Line; to which at the Point A let there be erected the Perpendicular AC. Let there be made the Angle (ACB) equal to the Angle (ACD) the accessible Line AB shall be equal to the inaccessible AD. Q. E. I.* Fig. 84.

PROP. XXVII. Theorem.

If the right Line GO shall cut two right Lines which are parallel (AB, CF); 1. The alternate Angles (RLO, QOL, likewise BLO, COL) shall be equal. 2. The external Angle GLB shall be equal to the internal one on the same Side (that is, to LOF;) as likewise Fig. 45.

G L R equal to L O C. 3. The two internal ones on the same Side (A L O, C O L) as taken together, shall be equal to two right ones, as likewise the two (B L O, F O L) equal to two right ones.

Fig. 46. The first Part is thus proved. From O and L draw the Perpendiculars O R, L Q. These are perpendicular to the * two Parallels, A B, C F; and by Definition 36, equal betwixt themselves, they shall therefore (*a*) intercept equal Parts of the Parallels, and R L shall be equal to Q O. Therefore the Triangles X and Z are equilateral to one another. Therefore (*b*) the alternate Angles R L O, Q O L which are opposite to the equal Sides R O, Q L are equal. Which is the first Thing. From whence it is likewise manifest that the Alternates B L O, C O L are equal. For because as well B L O, A L O as C O L, F O L are equal (*c*) to two right ones: therefore B L O, A L O together, are equal to C O L, F O L. Therefore taking away the Equals R L O, F O L, the remaining ones B L O, C O L shall be likewise equal.

* *Per, Ax.* 11.
(a) *Per Axio.* 12.
(b) *Per* 8. *l.* 1.
(c) *Per* 13. *l.* 1.

Part second. The Angle G L B is equal to that which is vertically opposite R L O (*d*); But R L O by the first Part of this Proposition is equal to L O F, Therefore G L B the external Angle is equal to the internal remote one which is on the same Side, L O F.

(d) *Per* 15. *l.* 1.

Part third. A L O by the first Part is equal to L O F. But L O F with C O L make Angles equal to two right ones. Therefore A L O with C O L doth the same.

Fig. 85. Coroll. *Hence in Imitation of* Eratosthenes *we learn to measure the Compass of the Earth. For he observed that on the Day of the Summer Solstice, the Sun was perpendicularly over* Siene, *a City of* Egypt; *and he found by the means of a Stile perpendicularly erected, that on the same Day the Sun was distant from the vertical Point of* Alexandria, *a City of* Egypt, *situate almost under the same Meridian with the other, seven Degrees, with one 5th Part of a Degree; and he knew that these two Cities were about* 5000 *Furlongs distant from each other. From these Things by the Help of this Proposition he determin'd the Compass of the Earth. Let A be* Siene, *and B be* Alexandria, *where the Gnomon B C is erected perpendicular to the Horizon. Let*

DF and EG be the Solar Rays parallel to one another as to Sense. DA a Ray perpendicular to the Horizon of Siene; *and EG a Ray Oblique to the Horizon of* Alexandria, *and which passing by the Top of the Gnomon makes with it the Angle GCF, which is of* $7\frac{1}{5}$ *Degrees. Now seeing the Angle GCF is equal to the alternate one AFB, and the measure of it is the Arch AB of* $7\frac{1}{5}$ *Degrees; he found the Compass of the Earth by this Analogy; as* $7\frac{1}{5}$ *Degrees are to 5000 Furlongs; so the whole Circumference, which is of 360 Degrees, is in a gross Number to 250000, the Compass of the Earth in the same Measure. Q. E. I.*

PROP. XXVIII. Theorem.

IF *a right Line (GO) cutting two right Lines (AB, CF) makes the alternate Angles (ALO, LOF) equal; the Lines (AB, FC) are parallel.* Fig. 47.

If you deny it, let XLZ passing thro' the Point L be parallel to CF. Therefore XLO (*a*) is equal to the alternate FOL, which cannot be, seeing by the Hypothesis ALO is equal to FOL.

(*a*) By the foregoing.

PROP. XXIX. Theorem.

IF *a right Line GO cutting two right Lines (AB, CF) shall make the external Angle (GLB) equal to the internal opposite one (LOF), or shall make the two internal Angles on the same Side (ALO, COL) equal to two right Angles; (AB, CF) are parallel Lines.* Fig. 45, & 46.

By the 15th of this Book GLB is equal to ALO, which is vertically opposite to it. But by the Hypothesis GLB is equal to LOF. Therefore also ALO is equal to its alternate one LOF. Therefore (*b*) AB, CF are parallel.

(*b*) By the foregoing.

Again, COL with FOL makes Angles equal to two right ones. But by the Hypothesis COL with ALO makes in all two right Angles also. Therefore ALO, FOL the alternate Angles are equal. Therefore, again, (*c*) AB, CF are parallel.

(*c*) By the foregoing.

Coroll. *From the second Part of this Proposition it appears that every Rectangle is a Parallelogram.*

PROP. XXX. Theorem.

Fig. 45. *If two right Lines (A B, C F) be parallel to the same right Line (D N) they are parallel betwixt themselves.*

It is manifest in it self, and from the foregoing Propositions. For if all be cut by the right Line G O, the external Angle G L B is equal (*a*) to the internal opposite one L D N. Now L D N is an external Angle in respect of D O F, and therefore (*b*) equal to it. Therefore also G L B is equal to L O F. Therefore A B, C F (*c*) are parallel.

(a) Per 27. l. 1.
(b) By the same.
(c) By the foregoing.

PROP. XXXI. Problem.

Fig. 48. *Thro' a given Point (A) to draw a Parallel to a given right Line (C F).*

From the Point A let there be drawn at random A L, cutting the given F C. At the Point A let there be made the Angle (*d*) L A S equal to the Angle A L F. The Line A S will be parallel to C F, as is manifest from the 28th, the alternate Angles S A L, A L F being equal.

(d) Per 22. l. 1.

As for the Practice. Draw A L, and from the Centre L describe an Arch I Q; and from the Centre A with the same Interval describe the Arch O X; from which having taken off O B equal to I Q; the right Line drawn thro' A and B will be the Parallel sought. The Demonstration depends upon the 29th, l. 1.

Fig. 49. Or otherwise thus. From a certain Centre P describe a Circle which may pass thro' the given Point A, and may cut the given Line C F in Q and O. Take the Arch O N equal to Q A. The right Line A N shall be the Parallel sought.

The Demonstration hereof depends upon 29. l. 3, and the 28th of this.

PROP.

PROP. XXXII. Theorem.

PART I.

IN every Triangle any one of the external Angles (as FBC) is equal to the two internal remote ones (A and C.) Fig. 51.

Thro' the Point B draw (a) B L parallel to A C. Because F A cuts the two Parallels B L, A C, the external Angle F B L shall be equal to the internal one A (b). And because the Line B C cuts the same Parallels (B L, A C); the Angle L B C shall be (c) equal to its alternate one C. Therefore the whole Angle F B C shall be equal to A and C both together. *Q. E. D.*

(a) Per. 31. l. 1.
(b) Per 27. l. 1
(c) By the same.

Corollaries.

1. THE external Angle F B C is greater than either of the internal opposite ones A or C. Fig. 51.

2. Of the Angles (C and A O B) having the same Base, A O B which falls within, is the greater. Fig. 39.

For let A O be produced unto F, A O B by this Proposition is greater than O F B; and likewise O F B is by this greater than C. Therefore A O B is much greater than C.

3. If from one Point A there falls two right Lines upon B C; one of them A O obliquely, the other A F perpendicularly; this last shall fall on the Side of the acute Angle A O B. For let it fall, if it may be, on the Side of the obtuse Angle A O C, as for Instance in Q. In this Case the acute Angle A O B shall be external in respect of A Q B, and consequently shall be greater than the right one, by Coroll. 1. which is absurd. Fig. 55.

PROP.

PROP. XXXII. Theorem.

PART II.

IN every Triangle the three Angles taken together are equal to two right ones, and therefore make 180 *Degrees.*

Fig. 52. (a) By the first Part of this. (b) Per 13. l. 1.

Draw forth one Side A B unto F. The external Angle F B C is equal (*a*) to the two internal opposite ones, A and C. But F B C with A B C make (*b*) Angles equal to two right ones. Therefore the two A and C with the same C B A make Angles equal to two right ones. *Q. E. D.*

Fig. 53. (c) Per 27. l. 1. (d) Coroll. 1. Prop. 13. l. 1.

Or thus. Draw the Line H M parallel to A C, the alternate Angles as well O and A, as N and C (*c*) are equal. But O, Q, N make Angles (*d*) equal to two right ones. Therefore also A, C, Q are equal to two right ones. *Q. E. D.*

Corollaries.

4. THE three Angles of any one Triangle taken together are equal to the three Angles of any other Triangle taken together.

5. If in a Triangle one Angle be right (or obtuse) the rest are acute.

6. If in a Triangle one Angle be right, the two other Angles together make one right Angle.

7. In every Triangle, the Angle which is right, is equal to the other two taken together.

8. When you know of how many Degrees one Angle of a Triangle is, you know at the same time how many Degrees the two other Angles as taken together do make up. And so on the contrary, when you know how many Degrees two Angles of a Triangle taken together do make up, or what is the Sum of them, you know at the same time of how many Degrees the third Angle is.

9. When two Angles of one Triangle either severally or together are equal, or two Angles of another Triangle, the third Angle of one Triangle is also equal to the third of the other.

10. When

10. When two Triangles have one equal Angle, the Sum alſo of the reſt of the Angles are equal.

11. When in an *Iſoſceles*, the Angle contained by the equal Sides is a right one, the two other are each of them half-right Angles. And the Angles of an *Iſoſceles* which are at the Baſe are always acute.

12. In an equilateral Triangle, each Angle is two thirds of a right Angle. For it is one third of two right ones, therefore it is two thirds of one right one.

13. Hence a right Angle (B A C) is eaſily divided into three equal Parts; if upon A C be made the equilateral Triangle Z; for ſeeing F A C is two thirds of one right one, B A F ſhall be one third of a right one. Fig. 54.

14. The Perpendicular A F is the ſhorteſt of all Lines which can be drawn from the Point (A) unto ſome right Line. For ſeeing the Angle F is a right one, A O F ſhall by Corollary the 5th be an acute one. Therefore (*a*) A F is ſhorter than any other, as A O. Fig. 55. (a) Per 19. l. 1.

15. Only one Perpendicular can fall from one Point unto one right Line. This is manifeſt out of the foregoing Corollary.

16. *Hence alſo we learn to determine the Parallax of the Stars, or the Difference of their true and apparent Place. Let A be the Centre of the Earth, B the Place of the Obſerver upon the Surface of the ſame. Let D B C be the Angle of the Star C according to Obſervation, or the* viſible *Angular Diſtance thereof from the vertical Point; when in the mean while D A C is the* true *angular Diſtance. Now the external Angle D B C which is given from Obſervation is equal to the Angles B A C and B C A taken together; and conſequently the Angle B C A is the difference of the Angles D B C and D A C. If therefore we ſhall from Aſtronomical Tables ſeek the Angle D A C, or what at that time of Obſervation is the true angular Diſtance of the Star from the vertical Point, when the Angle D B C is at the ſame time known by means of the Quadrant, the Difference of thoſe Angles B C A, which we call the* Parallax, *will likewiſe be known. Q. E. I.* Fig. 86.

Scholium.

BY the Teſtimony of *Eudemus* an antient Geometrician, *Pythagoras* was the Inventor of this Propoſition,

tion, which indeed is a Theorem most excellent in it self, most fruitful in its Consectaries, and of use in all Parts of the Mathematicks. *Aristotle* very frequently makes mention of it, who also puts it for an Example of the most perfect Demonstration. But like as from this Proposition we have already learned, how many right Angles the Angles of a Triangle are equivalent to; so by the help of the same, it will in the three following Propositions be manifest, how many right Angles the Angles of any rectilinear Figure whatsoever, whether internal or external, do make.

Theorem 1.

Fig. 56. IN every quadrangular Figure the four Angles together make four right ones.

For if thro' the opposite Angles you draw the right Line B F, this will cut the Quadrangle into two Triangles, without forming any new Angles, whose Angles together do (*a*) make four right Angles.

(a) Per 32. l. 1.

Theorem 2.

ALL the Angles together of every right-lin'd Figure make twice so many right ones, abating four, as are the sides of the Figure.

Fig. 57. From any Point A within the Figure let there be drawn unto the Angles of the Figure right Lines, which shall cut the Figure into so many Triangles as it hath Sides, and make no more Angles but those of the Centre. Wherefore when each of the Triangles contains two right Angles (*b*), they must altogether contain twice so many right Angles as there are Sides. Now the Angles about the Point A, (*c*) do make four right Angles. Therefore if from the Angles of all the Triangles you take away the new Angles which are about A, the remaining Angles which indeed do alone constitute the Angles of the Figure, will make twice so many right Angles, excepting four, as are the Sides of the Figure.

(b) Per 32. l. 1.

(c) Coroll. 3. Prop. 13. l. 1.

Hence it appears that all Right-lin'd Figures of the same Species, or Number of Sides and Angles, have the Sum of their Angles equal. Which thing is worthy of Admiration.

The

The Practice is thus; Double the Denominator of the Figure; and from the Product take away four; the Remainder is the Number of the right Angles, which the internal Angles of the Figure do make.

Theorem 3.

ALL the external Angles of any right-lin'd Figure whatsoever taken together do make up four right Angles. Fig. 58.

For each of the internal Angles of the Figure does (*d*) with its respective external one make two right Angles. Therefore all the internal ones, together with all the external ones, do make up twice so many right Angles as are the Sides of the Figure. Now by the Precedent, the internal ones, together with four right Angles added to them, make twice so many right Angles as are the Sides of the Figure. Therefore the external Angles are equal to four right ones. (d) Per 13. l. 1.

Wonderful truly is this Property of right-lin'd Figures; from whence it follows also, that all the right-lin'd Figures of any Species whatsoever have the Sums of their external Angles equal. And therefore the three external Angles of a Triangle are equal to the thousand external Angles of a thousand-sided Figure. Which Observation is altogether worthy of Admiration.

PROP. XXXIII. Theorem.

IF two right Lines, which are equal and parallel, as (A B, C F) be joined by two others (AC, B F); these also will be equal and parallel. Fig. 59.

Let A F cut the Parallels A B, C F. In the Triangles Q, R, the alternate Angles B A F, C F A (*a*) will be equal. Now the Side A B is supposed equal to the Side C F, and A F is common to both Triangles. Therefore (*b*) the Bases B F, A C are equal. (Which is the first Part.) And also the Angles at the Bases A F B, F A C are equal; so that A F falling upon the right Lines AC, and B F, makes the alternate Angles A F B, F A C equal. Therefore A C, B F are also (*c*) parallel. Which is the other Part. (a) Per 27. l. 1. (b) Per 4. l. 1. (c) Per 28. l. 1.

Coroll.

Fig. 57. Coroll. *Hence we learn to measure as well the Heights of Mountains above the Horizon as their horizontal Lines. Let ABC be the Side of a Mountain, to which apply a great Square, or some Instrument equivalent thereto ADB. Then shall AD be equal to HB, and DB equal to AH. Then coming unto the lower Part which is from the Point B unto the Point C, practise as before. So shall EB be equal to CF, and EC be equal to BF. Which done, the Sides parallel to the Horizon, AD, BE,* &c. *added together will give the horizontal Line GC; and the perpendicular Sides BD, EC,* &c. *added together will give the Height AG.*

Fig. 59. Coroll. (2.) *Hence also we learn to estimate the Composition of Motions. Let a Body placed at A be driven in the same Moment of Time by the Force AC according to the Direction of the Line AC, and by the Force AB according to the Direction of the Line AB. From the Conjunction of these two Forces it will describe the Diagonal AF. For in this Line of its Motion neither of the Forces is changed: For the Body at F is equally distant from both the Lines of Direction AC, AB, as if it had been driven by either of the Forces separately; which thing can be said of no other Point. And this Corollary doth so fully agree with Astronomical and other Mechanical Phænomena, that it is justly reckoned by the Famous Sir* Isaac Newton, *as a Foundation of his Geometrical Philosophy.*

PROP. XXXIV. Theorem.

Fig. 59. *In every Parallelogram the opposite Sides and Angles are equal, and it is cut into two equal Parts by the Diameter.*

(a) Per Def. 35. (b) Per 27. l. 1. (c) Per Def. 35. (d) Per 27. l. 1.

Because AB, CF are (*a*) parallel, and AF falls upon them, the alternate Angles BAF, CFA are (*b*) equal. Likewise because AC, BF (*c*) are parallel, and upon them falls the Line AF, the Alternates CAF, BFA (*d*) are equal. Therefore the whole Angle BAC is equal to the whole Angle BFC. In the same manner B and C are shewed to be equal. Which was the first Part.

Now

Now because it hath been already shew'd that the Triangles Q, R, which have one common Side AF, have also the Angles adjacent to the common Side equal, BAF to CFA; and CAF, to BFA; the Sides likewise shall be equal, AB to FC, and BF to AC; and thus the whole Triangles are equal. Which was the second Part.

Scholium.

PArtly from this Theorem, and partly from a Definition to be premis'd to the second Book, the measuring of a right-angled Parallelogram is easily deduced. The Area thereof being produced by the Multiplication Fig. 60. of the two contiguous Sides AF, AC one by another. *E.G.* Let AF be a Line of 8, AC a Line of 4 Feet. Multiply 8 by 4, there arises 32 Square Feet for the Area of the Rectangle.

But the Area of a Square is had from the Multiplication Fig. 61. of the Side FI by it self; as if FI be of 5 Feet, multiply 5 in it self, there will arise 25 square Feet for the Area of the Square.

The Demonstration is manifest from this Proposition, if parallel Lines be drawn thro' the Divisions of the Sides.

Corollary, *Hence Surveyors do easily divide the Area* Fig. 88. *of a Field when it is a Parallelogram. For let AB CD be the Parallelogram Field: AD the Diameter or Diagonal Line of the same, the middle Point whereof is marked F. Whatsoever right Line as EG, passeth thro' the Point F, it divides the Field into equal Parts EACG, EBDG. For the Triangle ABD is equal to the Triangle ACD, and * the Triangle AEF equal* * Per 26. l. 1. *to the Triangle GFD. If therefore to the Trapezium EBDF, instead of the Triangle AEF, you shall add the Triangle which is equal to it GFD, you will not change the Area; but the Trapezium EBDG will be equal to the Triangle ABD or to half the Parallelogram, and consequently equal to the Trapezium AE GC.* Q. E. I.

PROP.

PROP. XXXV, XXXVI. Theorems.

Fig. 62. P*Arallelograms upon the same or equal Bases (AB) and between the same Parallels (CQ, AX) are equal.*

(a) Per Def. 35. (b) Per 27. l. 1. (c) Per 34. l. 1. (d) Per 34. l. 1. (e) Per 4. l. 1.

Because AL, BQ (*a*) are parallel, and CQ cuts them, the external Angle CLA shall (*b*) be equal to the internal one FQB. Then because as well CF as LQ are equal (*c*) to the same AB, CF is equal to LQ. Add then FL to both, the whole Lines CL, FQ are equal. Moreover AL, BQ are equal (*d*). Therefore the Triangles CLA, FQB (*e*) are equal. Therefore taking away the common Triangle FOL, the Planes FOAC, QBOL remain equal: To each of which Trapeziums add the Triangle AOB, the whole Parallelograms ACFB, ALQB become equal. *Q. E. D.*

This Proposition will be made universal, *Prop.* 1. *l.* 6. Beginners may here observe, that, altho of two Parallelograms which are between the same Parallels infinitely produced, and upon the same Base, one of them be extended unto an infinite Length, it still remains but equal to the other, by the Force of the present Demonstration.

[*From hence it follows, that two Cities in Magnitude equal, may so much differ in Compass, that the Circumference of one may exceed that of the other an hundred or a thousand Times. If for Instance, one be of a Square Figure or Rectangular; but the other a Parallelogram, betwixt the same Parallels indeed with the former, but very oblong.*

Moreover, it hence follows, that Figures of equal Compass round may contain Area's vastly different.]

Scholium.

Fig. 62. FROM this Theorem we may learn to measure any Parallelogram. For the Area of it is produced from the perpendicular Altitude QX, or CA multiplied into the Base AB.

For

For the Area of the Rectangle CB, which is equal to that of the Parallelogram B, L, is made (*a*) by A C, multiplying A B. Therefore, &c. (a) By the foregoing Scholium.

PROP. XXXVII, XXXVIII. Theorem.

Triangles (ACB, ALB) upon the same or equal Bases (AB), and between the same Parallels (CI, AZ) are equal. Fig. 63.

Draw the Lines B L, B I parallel to the Sides A C, A L. The Parallelograms ACFB, ALIB (*b*) are equal. But the given Triangles are halves of those Parallelograms (*c*). Therefore the given Triangles (*d*) are equal. (b) By the foregoing. (c) Per 34. l. 1. (d) Per Axio. 6.

This Proposition will be made universal, *Prop.* 1. *l.* 6. Let Beginners mark the same Thing here concerning Triangles, which we bid them to note in the foregoing Proposition concerning Parallelograms.

Coroll. (1.) *Hence Surveyors easily divide the Area of a triangular Field. Let ABC be the Field, and let the Base BC be bisected in D. The Triangles ABD, ADC upon the equal Bases BD and DC, and having a common Top A, or being between the same Parallels, are equal.* Q. E. F. Fig. 89.

[Coroll. (2.) *Hence we also gather, with the famous Sir* Isaac Newton, *that the Area's which all Bodies whatsoever that revolve round about an immoveable Centre, towards which they are impell'd, do describe, are both in immoveable Planes, and are proportional to the times of Description. For let the Time be divided into equal Parts, and in the first equal Part of Time, let the Body by the impress'd Force describe the right Line AB. The same Body in the second Part of Time, if nothing hindred, would go forward strait unto c, describing the Line Bc equal to AB; so that the Area's made by Lines drawn from the Centre ASB, BSc* (*e*) *would be equal. But when the Body comes unto B, let the Force act with one single Impulse, but a great one, and make the Body to deflect from Bc, and to go forwards in the right Line BC;* i. e. *let the centripetal Force be in that Place to the Force before impuls'd, as Cc or Bg is to Bc; in this Case the Body will* Fig. 90. (e) Per 37. l. 1.

(a) Per Cor. 2 Prop. 33. l. 1. *(a) describe the Diagonal B C. Let there be drawn parallel to B S the right Line C c meeting B C in C. In the second Part of Time compleated, the Body will be found in the Point C in the same Plane with the first Triangle S A B. Join S C. The Area made by a Ray drawn from the Centre, that is the Triangle S B C, will be equal to (b) S B c, and consequently to the first Triangle S A B (c). By the same Argument the Body in the third equal Part of Time would by its present Force reach from C unto d, so that the Line C d should be equal to the Line B c or A B. But if the centripetal Force, whether it be greater or lesser, does again act upon it in the Point C, in the end of the third Part of Time, it will be found somewhere in the Line D d, parallel to S C; and therefore as before, supposing the said Force to be equal or unequal to what it was before, it will be found to have described the Diagonal C D, and will be found in the Point D; and a Ray being drawn from the Centre, the Triangle S D C will be equal to that S d C, and consequently to the others S C B, S A B, which are equal one to the other. In like manner, if the centripetal Force act successively in the Points D, E, F, and be the cause that the Body in the several Parts of time respectively describes the Diagonals, D E, E F, &c. the Area's now made as afore will be in the same Plain, and Triangles will be described equal to the former Triangles. Therefore in equal Times equal Area's are described in an immoveable Plane; and so the Sums of the Area's S A D S, S A F S will be amongst themselves as the Times wherein they were described. Now let the Number of the Triangles be increased, and their Wideness diminished infinitely; both that last Perimeter of them A B C D E F, will be a Curve Line, and the Area's described in one and the same immoveable Plane will in this Case also be proportional to the Times as well as before.* Q. E. D.]

(b) Per 37. l. 1.

(c) Per Axio. 1.

PROP. XXXIX, XL. Theorems.

Fig. 64.

EQual Triangles (*A C B, A F B*) *upon the same or an equal Base* (*A B*) *and on the same Side, are between the same Parallels,* (*A B, C F*).

If you deny it, let CL be parallel to AB, and let BL be drawn. Then ALB is equal to ACB (*a*). But by the Hypothesis AFB is equal to ACB. Therefore ALB and AFB are equal; *i.e.* a Part is equal to the whole. Which cannot be. Therefore, &c. (a) By the foregoing.

[Corol. (1.) *Hence also, with the famous Sir* Isaac Newton, *we gather, that all Bodies which are moved in Curve Lines, and describe Area's about some Centre proportional to the Times, are perpetually urged and press'd by a Force impelling towards the Centre. For because of the Equality of the Triangles SCB, ScB described upon the same Base SB, the Points C and c shall be in a Line Cc which is parallel to the Base; and so the Figure BcCg shall be a Parallelogram; the Sides whereof Bc and Bg are * the Lines of the Directions of the Forces; and BC is the Diagonal. The Body therefore is urged unto C by the Force Bg, which tends unto S the Centre. And so in all the Points, C, D, E, F.* Q. E. D. * Per Coroll. 2. Prop. 33. l. 1.

Corol. (2.) *Seeing therefore in the Motion of the primary Planets, the Area's made by Rays, or right Lines drawn from them unto the Sun, are always proportional to the Times, as all Astronomers know, the Planets are urged by a perpetual Force, which tends to the Sun. And the same thing is equally true of the secondary Planets with respect to their primary ones.*]

PROP. XLI. Theorem.

IF a Triangle (AFB) be in the same Parallels with a Parallelogram (AL) and have the same or an equal Base (AB) it is half of the Parallelogram. Fig. 64.

Draw CB. The Triangles AFB, ACB are (*b*) equal. But ACB is half of the Parallelogram AL (*c*). Therefore AFB also is half of AL. *Q. E. D.* (b) Per 37, 38. l. 1. (c) Per 34. l. 1.

Scholium.

FRom this Proposition, with the Schollium of Prop. 35. we learn that the Area of whatsoever Triangle, as AFB, is produced from half the Altitude FI multiplied into the Base AB, or half the Base multiplied Fig. 65.

 into

into the Altitude. Wherefore one Side of a Triangle being known, and the Height, that is, the Perpendicular which falls upon the known Side from the opposite Angle, the Measure of the Triangle is given. As if the Base A B be of an 100 Feet, the Height F I, 85, multiply half the Base 50 by 85, and you have the Area of the Triangle A F B=4250 Feet Square. Further, the Altitude of a Triangle, when the Area of it is in all Points accessible, may be known mechanically as well as the Sides. But if the Area of it cannot be gone over, the Height may be found Geometrically by 12 and 13. *lib.* 2. as we shall there shew.

In a rectangle Triangle, the Height is the same with either of the Sides about the right Angle. Half of this therefore multiplied into the other Side adjacent to the right Angle, will give the Area of the Triangle.

PROP. XLII. Problem.

Fig. 66. *To make a Parallelogram with an Angle equal to a given one (O); and equal to a given Triangle (A B C.)*

Bisect the Base A B in F. Thro' C draw C X parallel (*a*) to AB. Make the Angle B A L equal to the given one O (*b*). Draw F I parallel (*c*) to A L. A L I F shall be that which was sought for.

For let F C be drawn. The Parallelogram A I hath an Angle L A F equal to the given one O, and is equal to the given Triangle A C B; since, as well the Triangle A C B (*d*) as the Parallelogram A I (*e*) is double to the same Triangle A C F.

(a) *Per* 31. *l.* 1.
(b) *Per* 23. *l.* 1.
(c) *Per* 31. *l.* 1.
(d) *Per* 38. *l.* 1.
(e) By the foregoing.

Corollary.

Fig. 66. THE Triangle A C B being given, a Rectangle equal to it is had, if there be drawn a Line parallel to the Side A B, and A B being bisected in F, the Perpendicular B Q be erected. For the Rectangle under F B and Q B will be equal to the Triangle A C B.

PROP.

PROP. XLIII. Theorem.

IN a Parallelogram (as B L) the Complements (B O, O L) of those Parallelograms which are about the Diameter (R F, C S) are equal. Fig. 67.

If thro' any Point of the Diameter A Q, as the Point O, C F be drawn parallel to the Side A B, and R S parallel to the Side B Q; the whole Parallelogram B L is divided into four Parallelograms, whereof two are about the Diameter R F, C S, the other two B O, O L are the Complements of these unto the whole Parallelogram B L.

Their Equality is thus proved. The Triangles (*a*) A B Q, A L Q are equal. Likewise the Triangles A R O, O C Q (*b*) are equal to the Triangles A F O, O S Q. Therefore if from the Equals (*c*) A B Q, A L Q, you take away Equals, on this Side A R O, O C Q, on that A F O, O S Q; then B O and O L shall remain equal. *Q. E. D.*

(a) Per 34. l. 1.
(b) By the same.
(c) Per Ax. 3.

PROP. XLIV. Problem.

UPON a given right Line (O S) to constitute a Parallelogram, in a given Angle (X,) which Parallelogram shall be equal to a given Triangle (V). Fig. 68.

Make a Parallelogram (*d*) R C equal to the given V, having its Angle R O C equal to the given one X, and join the Side R O directly to the given Line O S, so as to make one right Line therewith. Then thro' S draw S Q (*e*) parallel to O C, which S Q let B C meet when it is produced unto Q. Then let a right Line drawn thro' Q and O meet B R produced unto A. Which done, thro' A draw A L parallel to O S, which A L let C O and Q S meet when it is produc'd unto F and L; the Parallelogram O L is that which was required.

For O L (*f*) is equal to R C, that is, by the Construction, to the given Triangle V, and is at the given

(d) Per 42. l. 1.
(e) Per 31. l. 1.
(f) By the foregoing.

(a) Per 15. l. 1. Line OS; and (a) the Angle FOS is equal to the Angle ROC; that is, by the Construction, equal to the given Angle X.

Scholium. *This Proposition contains a certain Geometrical Division. For in the vulgar Arithmetical Division, the Number to be divided may justly be considered as being a certain Rectangle.* e. g. *Let the Rectangle AB comprehending* 12 *square Feet, be to be divided by* 2; i. e. *a Rectangle is to be found equal to that AB of* 12 *square Feet, one of whose Sides shall be only* 2 *Feet: From whence it comes to be enquired of what Number the Side sought shall consist; which Side is to be esteemed a certain Quotient of this Division. Which Thing is performed Geometrically after this manner: With a Pair of Compasses take the Line BD of two Feet, and draw the Diagonal DEF. The Line AF is that which is sought for. For the Complements EG and EC are* * *equal; and in the Rectangle EG one Side EH is equal to the Line BD which is of* 2 *Feet; and the Side EI is equal to AF.*

Fig. 91.

* Per 43. l. 1.

This kind of Division is called Application, *because the rectangular Space AB is* Applied *to the Line BD or EH: and hence it comes, that Division is often named* Application; *respect being had to the Practice of the old Geometricians, who always made more use of Geometrical Construction, which requires only a Rule and a pair of Compasses, than of Arithmetical Computation, which is performed by Number.*

PROP. XLV. Problem.

Fig. 69. UPON *a given Line (IQ) and in a given Angle (H) to make a Parallelogram equal to a given Rectilinear Figure (CBA).*

Resolve the given Rectilinear into the Triangles A, B, C, by drawing the right Lines FL, FI.

Upon the given Line IQ in the given Angle H make (b) the Parallelogram IV equal to the Triangle A. Then the right Line IR being produced infinitely towards P; upon the right Line RV in the Angle VRP (c) make the Parallelogram RZ equal to the Triangle

(b) Per 44. l. 1.

(c) By the same.

angle

angle B. Again, upon the Line SZ with the Angle ZSP make the Parallelogram SG equal to the Triangle C. This done, I say IG is the Parallelogram sought for.

For (*a*) the Angle ZVR is equal to its alternate IRV. But (*b*) QVR and IRV are equal to two right Angles. Therefore also QVR and ZVR are equal to two right ones. Therefore * QV and ZV fall directly so as to make one right Line. After the same manner I might shew that QZ and ZG make one right Line. Therefore the whole QVZG is one right Line, and is also parallel to IX, seeing by the Construction QV is parallel to IP. Now XG also (*c*) is parallel to IQ; seeing XG is parallel to SZ, and SZ to RV, and RV to IQ.

(a) Per 27. l. 1.
(b) By the same.
* Per 14. l. 1.
(c) Per 30. l. 1.

IG therefore (*d*) is a Parallelogram; but that it is such an one as was required, is manifest from the Construction.

(d) Per def. 35.

[Coroll. *Hence is easily found the Excess whereby a greater rectilinear Figure exceeds a lesser: To wit, if unto the same right Line IQ be applied Parallelograms respectively equal to the two right-lin'd Figures. For that Parallelogram by which the greater Rectilinear exceeds the lesser, will give the difference of them.* Q. E. I.]

Scholium.

WE will here add a Problem that will be useful for the Practice of Proposition XIV. l. 2.

A quadrangular Figure BF being given, to describe an equal Rectangle.

Fig. 70.

Resolve it into Triangles by the right Line AC. From the opposite Angles let down the Perpendiculars BO, FI. Bisect AC in S. From S erect the Perpendicular SL, equal to the two BO, FI put together. The Rectangle comprehended under LS and SC is equal to the given BF. The Demonstration appears out of Proposition XLI.

PROP. XLVI. Problem.

Fig. 71.

FROM a given right Line (A B) to describe a Square.

Erect two Perpendiculars equal to the given A B; to wit, A C, B E, then join C E. I say the Thing is done. For seeing the two Angles A and B are (*a*) right ones, A C and B E shall (*b*) be parallel; but they are also (*c*) equal. Therefore C E and A B are (*d*) parallel and equal. Therefore the Figure is Parallelogram and Equilateral. But all the Angles also are right ones (for seeing A and B are right Angles, the opposite ones (*e*) E and C are right also.) Therefore the Figure A E is a Square.

(a) By the construction
(b) Per 29. l. 1.
(c) By the construction
(d) Per 33. l. 1.
(e) Per 34. l. 1.

[*In the same manner you may easily describe a Rectangle which hath the two unequal Sides given.*]

PROP. XLVII. Theorem.

Fig. 72.

IN every right-angled Triangle (as A B C) the Square of the Side (A C) which is opposite to the right Angle, is equal to the two Squares together of the two other Sides (A B, C B.)

Let I C and B F be drawn; and B E parallel to A F. Now if to the right and therefore equal Angles I A B, F A C, there be added the common Angle B A C, the Wholes I A C and F A B shall be equal. But in the Triangles I A C, F A B, the Sides which contain those equal Angles are equal (*f*) amongst themselves, to wit, I A, C A, to B A, A F, each to each. Therefore the Triangles I A C, F A B (*g*) are equal. Which because they stand upon the same Bases I A, F A with the Parallelograms A B L I and Z A F E, and between the same Parallels I A, L B C, and A F, E Z B, they are halves (*h*) of those Parallelograms. Therefore the Parallelograms A B L I, Z A F E, as being Doubles of Equals, are equal betwixt themselves. By the same reasoning if right Lines A X, B R were drawn, it might be shewn

(f) Per Def. Square.
(g) Per 4. l. 1.
(h) Per 41. l. 1.

that the Parallelograms E C, B X are equal. Therefore the whole A R is equal to I B and B X together. *Q. E. D.*

It was taken for granted that L B C is parallel to I A, in order to which L B and B C must be one right Line. Now that they are so is manifest from the 14th, seeing the Angles L B A and C B A are both right ones by the Hypothesis.

Scholium.

THIS Theorem (which *Prop.* 31. l. 6. *Euclid* extends unto all like or similar Figures) is commonly call'd the *Pythageric* Theorem, from *Pythagoras* the Inventor of it; who, as is attested by *Proclus*, *Vitruvius* and others, offer'd Sacrifices to the Muses, as supposing himself to have been helped by them in so excellent an Invention; in which thing he shew'd himself to be ignorant of God, the Lord of Sciences, the true and only Author of all Wisdom; or certainly if he knew him, he glorified him not as God. There is frequent and notable Use of this Theorem thro' all the Mathematicks; and in particular it opens a way unto the Knowledge of incommensurable Magnitudes, a main Secret of Geometrical Philosophy.

That the Side of a Square is incommensurable to the Diameter, is a thing much celebrated amongst the old Philosophers, *Aristotle* and *Plato* especially; insomuch that *Plato* would say, that he who knows not this is not a Man, but a Beast. Now the Knowledge of this Mystery seems to have taken its Rise out of this 47th Proposition. For seeing in the Square A E the Angle A is a right Angle, the Square of the Diameter C B shall be equal to both the Squares of the Sides, A B, AC, and therefore double to one of them. Wherefore seeing the Square of C B is 2, and the Square of the Side A B is 1 or Unity, the Diameter C B shall be the square Root of 2, and the Side A B the square Root of Unity, or Unity it self; the *Ratio* of which Quantities (as it will be demonstrated in its Place) cannot be explicated in Numbers, and therefore they are incommensurable. *Fig.* 71.

And by this one Argument alone, if all others were wanting, it might evidently be made out, that Geometrical Magnitudes cannot be made up of a definite Number

ber of Points; for otherwise none would be incommensurable; forasmuch as a Point would be the common Measure of all.

To these Things we will subjoin three Problems which are deduced out of the present Proposition, and are of frequent Use.

Problem 1.

Fig. 73. IF any Number of Squares are given, to make one equal to them all together.

Let there be three or more Squares given, whose Sides are AB, BC, CE. Make the right Angle FBZ having indefinite Sides, and unto the Sides of it transfer AB and BC, and then join AC. The Square of AC shall be equal to the Squares of AB, and BC together (a). Then transfer AC from B unto X, and CE the third given Side transfer from B unto E, and join EX; the Square of EX shall be equal to the Squares of EB (or EC) and BX together; that is, equal to the three given Squares, whose Sides are AB, BC, CE: And so on as long as you please.

(a) Per 47. l. 1.

Problem 2.

Fig. 74. TWO unequal right Lines being given (AB, BC) to determine that Square, whereby the Square of the greater (AB) exceeds the Square of the less (BC.)

From the Centre B with the Interval AB describe a Circle. Then from C erect a Perpendicular CE, cutting the Circumference in E. The Square of CE is the Excess or Difference which is sought for.

For let BE be drawn. The Square of BE, that is, of AB, is equal to the Squares (b) of BC and CE together. Therefore, &c.

(b) Per 47. l. 1.

Problem 3.

Fig. 75. ANY two Sides of a right-angled Triangle being known, to find the third.

Let the Sides containing the right Angle be AB, AC, the

the one of 6 Feet the other of 8. You are to find of how many Feet the Side CB, which is opposite to the right Angle, is. To do which, multiply 6 and 8 each of them by it self. From which Multiplication there will arise for the Squares of those two Sides 36 and 64; the Sum of which is 100. The square Root of 100, which is 10, gives the Feet of the Side BC, whose Quantity was sought. This Demonstration offers it self in and from this 47th Proposition, for the Sum of the Squares BA and CA is equal to the Square of BC. Therefore the Root of the Sum of them is equal to the Root or Side BC.

Then let the Sides AB, BC be known, the one of 6 Feet, the other of 10, you are now to find AC. Take the Square of the Side AB which is 36, out of the Square of the Side BC=100. The Remainder 64 shall be the Square of the Side AC. The Root therefore of 64, which is 8, gives the Feet of the Side AC.

Corollary. *From hence we derive the Original of the Tables of Sines, Tangents, and Secants. For, for Instance, let AC the Semidiameter of the Circle be of* 100,000 *Parts, and the Angle BAD of* 30 *Degrees. Because the Chord or Subtense of* 60 *Degrees * is equal to AC the Semidiameter; BD the Sine of* 30 *Degrees shall be equal to half the Semidiameter, or* $\frac{1}{2}$ *AC; and therefore shall contain* 50,000 *Parts. But now in the right-angled Triangle ADB, the Square of AB is equal to the Squares of AD and BD. Wherefore let the Semidiameter AB be squared (by multiplying* 100,000 *by* 100,000*) and from that Square subtract the Square of BD. The Remainder shall be the Square of AD, or of the Cosine equal to it BF; out of which extract the square Root, and you will have the Line BF or AD. Then by this following Analogy, AB : BD : : AE : CE, or AD : BD : : AC : CE, will be had the Tangent CE. And then lastly, if the Square of AC be added to the Square of CE, the Root of the Sum being extracted will be the Secant AE,* Q. E. I,

Fig. 92.

* Per Corol. 1. Prop. 15. l. 4. and Corol. 2. Prop. 3. l. 3.

PROP. XLVIII. Theorem.

Fig. 76.

IF in a Triangle the Square of one of the Sides (A B) be equal to the two Squares of the other Sides (A C, B C) taken together, the Angle (A C B) which the two other Sides contain, is a right Angle.

If not, the Angle ACB will be greater or less than a right Angle. In either of which Cases (as it will be demonstrated, *Prop.* 12, 13. *l.* 2, which Propositions depend not on this) the Square of AB will not be equal to the Squares of AC, BC together; which is contrary to the Hypothesis.

Or thus. Draw FC perpendicular to AC, and equal to CB, and join AF. The Square of AF is *(a)* equal to the Squares of FC, CA together; that is, *(b)* to the Squares of BC, CA; that is, by the Hypothesis, to the Square of AB. Therefore the right Lines AF, AB are equal. Because therefore the Triangles are mutually equilateral, the Angles at C *(c)* are equal. Therefore they are both right Angles *(d)*. *Q. E. D.*

(a) Per 47. l. 1.
(b) By the construction.
(c) Per 8. l. 1.
(d) Per def. 14.

THE

the Right Lines A B and D

Fig. 76.

(a) *Per* 47. l. 1.
(b) By the construction

(c) *Per* 8. l. 1.
(d) *Per def.* 14.

The Elements of EUCLID.

BOOK II.

THIS Book is small indeed in Bulk, but great in the Excellence and Usefulness of its Theorems. Young Beginners will not, I know what I say, be at first able to comprehend it; but being further advanced, they will, from their own Experience, find that it is most certainly true.

A DEFINITION.

A Right-angled Parallelogram (as A E), (which is wont simply, and without any Addition, to be call'd a Rectangle) is said to be contain'd under the two Lines (A C, A F) which determine the Magnitude of it: Fig. 60. l. 1.

For the one of them A C determines the Height, the other A F the Breadth of it. Now if the Side A C be understood to be carried perpendicularly along the whole A F, or A F along A C, by that Motion the Rectangle or its Area will be produc'd. Wherefore a Rectangle is rightly said to be produced from the drawing of two Lines into one another, or the Multiplication of them one by the other. When therefore you have these Words, [the Rectangle under (or of) A C, C B] or for Brevity's sake, [the Rectangle A C B] there is meant that Rectangle which is contain'd under A C and C B multiply'd one into the other. In like manner, when we say the Rectangle under AB, B C, or the Rectangle A B C, there is design'd the Rectangle contain'd under the Right Lines A B and B C, multiply'd by one another. Fig. 2. l. 2.

More-

Moreover, of Rectangles some are Oblong, some are Square. The Oblong Rectangle is that which hath its contiguous Sides unequal, or which is contained under two unequal right Lines. The square Rectangle that which is contain'd under two equal right Lines.

PROPOSITION I. Theorem.

Fig. 1. l. 2. *IF there be two right Lines (AB, AC), one whereof is divided into as many Parts as you will (AE, EF, FC); the Rectangle compriz'd under those two (AB, AC) is equal to all the Rectangles together, which are contain'd under the undivided Line (AB) and the several Parts of the divided Line (AE, EF, FC).*

Make AB perpendicular to AC, thro' B draw the infinite Line BR parallel to AC. From E, F, C, erect the Perpendiculars EI, FL, CQ. BC will be a Rectangle under AB and AC; and is equal to the Rectangles BE, IF, LC; that is, (because as well IE as LF are equal * to AB) equal to the Rectangles under AB, AE; AB, EF; AB, FC.

* Per 29, & 34. l. 1.

Scholium.

THE ten first Theorems of this Book are true also in Numbers, if they as Lines be divided into Parts. The numerical Rectangles are produced from the Multiplication of two Numbers, and the numerical Squares from the Multiplication of the same Number by it self.

[*Let the undivided Number be* 9, *and the divided one* 12. *The Rectangle which is from* 9 *multiplying* 12=108 *will be equal to the three Rectangles,* 27, 36, *and* 45, *which are produced from* 9 *multiplied by* 3, *and* 4 *and* 5, *respectively and separately. Or let the Number* 432 *be as it were a Multiplicand divided into* 400 *and* 30 *and* 2; *and the Number* 8, *an undivided Multiplier;* $8 \times 432 = 3456$ *will be equal to* $8 \times 400 = 3200 + 8 \times 30 = 240 + 8 \times 2 = 16$. *And from this Proposition therefore the Demonstration of Multiplication is to be deriv'd.*]

PROP.

PROP. II. Theorem.

IF the right Line (AB) be cut any where (as in C), two Rectangles under the whole (AB) and the Parts (AC, CB) are equal to the Square of the whole Line (AB.) Fig. 2.

[*For AD is the Square of the whole; and AH, CD are Rectangles under the Whole AB, and the Parts AC, CB.* Fig. 17.

Let the Number 8 *be divided into* 5 *and* 3; *the Square of the Whole* $8\times8=64$, *is equal to the Rectangles* $8\times3=24$, $+8\times5=40$.]

PROP. III. Theorem.

LET a right Line (as AB) be cut any where (as for Instance in C); the Rectangle contain'd under the whole AB, and either of the Parts (BC) is equal to the Rectangle under the Parts (AC, CB) together with the Square of the said Part (BC.) Fig. 3.

[*For AF is the Rectangle under the whole Line AB, and the Part AC; and CF is the Rectangle under the Parts; as AE is the Square of the Part AC.* Fig. 18.

In Numbers. Let the Number 7 *be divided into the Parts* 3 *and* 4. *The Rectangle of* $7\times3=21$ *is equal to the Rectangle of* $3\times4=12$, *together with the Square* $3\times3=9$. *In like manner* $7\times4=28$, *is equal to the Rectangle* $3\times4=12+$ *the Square* $4\times4=16$.]

PROP. IV. Theorem.

LET a right Line, (as FL) be cut any where, (as in O): The Square of the whole shall be equal to the Squares of the Parts (FO, OL) and to two Rectangles contain'd under the Parts (FO, OL). Fig. 4.

[*For FD is the Square of the whole; and CG and CL the Squares of the Parts; and CF, CD, two Rectangles under the Parts.* Fig. 19.

In

In Numbers. Let the Number 10 *be divided into two Parts* 7 *and* 3. *The Square of* $10 \times 10 = 100$ *is equal to the Squares of the Parts* $7 \times 7 = 49$, *and* $3 \times 3 = 9$, *and to the two Rectangles* $7 \times 3 = 21$, *and* $7 \times 3 = 21$. *And on this Proposition depends the Extraction of the Square Root.*

Fig. 19. Coroll. *(1.) Hence it is manifest, that the Parallelograms about the Diameter of a Square, (O I, H K) are Squares.*

(2.) As likewise that the Diameter of every Square bisects the Angles of it.

(3.) And that the Square of half of a Line is a fourth Part of the Square of the whole Line. For in that Case the Rectangles and Squares end in four equal Squares.]

PROP. V. Theorem.

Fig. 5. *IF a right Line (as Q X) be cut equally in (R), and unequally in (S), the Rectangle contain'd under the unequal Parts (Q S, S X) taken together with the Square of the intermediate Part (R S) shall be equal to the Square of the half (Q R).*

Fig. 20. [*For Q H is the Rectangle under the unequal Parts, and L G the Square of the intermediate Part, and R F the Square of half the Line; and therefore, because the Rectangle Q L is equal to the Rectangle S F, and the rest of the Space is common to both, the Proposition is manifest.*

Let the Number 8 *be divided equally, that is, into* 4 *and* 4, *and unequally into* 5 *and* 3. *The Rectangle of* $5 \times 3 = 15$ *together with the Square* $1 \times 1 = 1$ *shall be equal to the Square* $4 \times 4 = 16$.]

PROP. VI. Theorem.

Fig. 6. *IF a right Line (A B) be divided into two equal Parts in (C), and to it a certain right Line (B F) be adjoin'd; the Rectangle contain'd under the whole compound Line (A F) and the adjoin'd one (B F) taken together with the Square of half the Line (C B) shall be*

be equal to the Square of (CF) which is compounded of half the Line AB, and the adjoin'd one.

[*For AN is the Rectangle under the whole compound Line and the adjoin'd one; and KG the Square of half the Line AB; and CE the Square of the Line compounded of half the Line AB, and that which was added. Wherefore because the Rectangle HE is equal to the Rectangle AK, and the rest of the Space is common to both, AN and KG is equal to CE.* Q. E. D.] Fig. 21.

[*If the Number 6 be divided into the two equal Parts* 3 *and* 3; *and to it be added the Number* 2; *The Rectangle of* $8\times2=16$, *taken together with the Square* $3\times3=9$, *shall be equal to the Square* $5\times5=25$.]

Coroll. *Hence, with* Maurolycus, *with one single Observation we learn to measure the Diameter of the Earth. Let the Altitude of the Mountain AD be known, and AB the Line touching the Earth be known by measuring. Let the Line DE be cut into two equal Parts in the Centre C, and to it be added the Line AD. Now because the Rectangle under AE, AD, together with the Square of DC, is by this Proposition equal to the Square of AC, that is, equal to the * Squares of the Lines AB, BC; from hence it follows, that if you take away on both sides the Square of CD or CB, the Rectangle which is under AE, AD is equal to the Square of AB. Therefore let the known Square of AB be divided by the known Altitude of the Mountain AD, and the Quotient will give the Line AE. From which subtract the known Altitude of the Mountain AD, the remaining Line DE will be the Diameter of the Earth.* Q. E. I. Fig. 23. * Per 17. l. 1.

PROP VII. Theorem.

IF a right Line (AB) be cut any where (as in C), the Square of the whole Line (AB) taken together with the Square of either of the Segments (AC) is equal to two Rectangles contained under the whole (AB), and that Segment (AC), together with the Square of the other Segment (CB). Fig. 7.

Fig. 23. [*For EB is the Square of the whole Line, and AL the Square of the Part AC. But the two Rectangles under the whole Line and that Part, EI, HL, together with GB the Square of the other Part, possess the same Space that EB and the Square of AC doth. Therefore they are equal to EB and the Square of AC.*]

[*Let the Number* 13 *be divided into any two Parts, as* 9 *and* 4. *The Square* $13\times13=169$, *together with that* $9\times9=81$, *is equal to* $13\times9=117$, *and* $13\times9=117$, *and the Square* $4\times4=16$.]

PROP. VIII. Theorem.

Fig. 8. *If a right Line (LF) be divided into two equal Parts in (I), and to it a certain right Line be adjoin'd (FO); the Rectangle (LIO) which is contain'd under the half of the Line (LI), and the Line (IO) that is compounded of half the aforesaid Line, and the Line adjoin'd; this Rectangle (I say) taken four times, together with the Square of the adjoin'd Line (FO), shall be equal to the Square of the whole compound Line (LO.)*

Fig. 24. [*For AL is the Square of the whole Compound, containing four equal Rectangles under LI and IO (to wit, DR, BQ, RO, and the fourth made up of LR and QH added together) and with those four Rectangles the Square HE. From whence the Proposition is manifest.*]

[*Let the Number* 12 *be divided into* 6 *and* 6; *and the Number* 4 *be added to it. The four Rectangles* $10\times6=240$ *and* $4\times4=16$ *are equal to the Square* $16\times16=256$.]

PROP. IX. Theorem.

Fig. 9. *If a right Line (AC) be divided equally in (B) and unequally in (F), the Squares of the unequal Parts (AF, FC) will be double to the Squares of half the Line (AB), and of the intermediate Part (BF.)*

Fig. 25. [*Let BE be equal and perpendicular to BA. From hence the Construction being made, as the Figure shews,*

*the Lines AB, BE, BC will be equal: As also the Lines EG, GQ will be equal. The Angles AEC, ABE, CBE, EGQ, QFC, will be equal; and the Angles AEB, BEC, ECA, CQF, EQG half right ones. From whence the Square of AE will be double to * the Square of AB, which is half of AC; and the Square of EQ double to the Square of GQ or BF the intermediate Line. But the Squares of AE and EQ are † equal to the Square of AQ, that is, to the Squares of AF and FQ or FC the unequal Parts.* Q.E.D.] * Per 47. l. 1. † By the same.

[*Let the Number* 32 *be divided equally into* 16 *and* 16, *and unequally into* 20 *and* 12. *The Square* 20 × 20 = 400 *with the Square* 12 × 12 = 144, *are double to the Squares of* 16 × 16 = 256 *and* 4 × 4 = 16.]

PROP. X. Theorem.

IF a right Line (FI) be divided into two equal Parts in (L), and to it a certain right Line (as IO) be adjoin'd; the Square of the whole compound Line (FO), taken together with the Square of the additional Line (IO), shall be double to the Squares, which are described upon the half Line (FL), and (LO) that which is compounded of half the Line (FI) and the additional Line. Fig. 10.

[*For a Construction being supposed not unlike to the former; the Square of FE, is double to the Square * of the half Line FL, and the Square of EG is double to the * Square of EQ or LO, which is compounded of the half Line and the additional one. But the Squares of FE and EG are equal to the * Square FG; that is, to the Square of FO the whole compound Line; taken together with the Square of OG or OI the additional Line.* Q.E.D. Fig. 26. * Per 47. l. 1.

Let the Number 40 *be divided into* 20 *and* 20, *and to it let there be added the Number* 14. *The Square* 54 × 54 = 2916, *with the Square* 14 × 14 = 196, *are double to the Square of* 20 × 20 = 400, *taken together with* 34 × 34 = 1156.]

PROP. XI. Problem.

Fig. 11.

TO cut the given right Line (AB) in (C) so, that the Rectangle (ABC) which is contain'd under the whole Line and one Part, shall be equal to the Square of the other Part (AC).

From A erect a perpendicular AF equal to AB. Bisect AF in X. Draw the right Line XB; from the Line FA produc'd, cut off XI equal to XB. Then cut off AC equal to AI. I say the Thing is done.

For let the Square BAFS be perfected; and a Perpendicular being drawn thro' C, let the Rectangle FILO be perfected also. Because FA is bisected in X, and to it is added AI; there shall be

(a) Per 6. 2.

$$\left.\begin{array}{l}\text{the Rect. FIA}\\ \quad + \\ \text{Square of XA}\end{array}\right\} = (a) \text{ to the Square of XI}$$

(b) By the Construction.

That is, = to the Square of XB (*b*)

(c) Per 47. l. 1.

$$\text{That is, = to the Squares of } \left.\begin{array}{l}\text{AB}\\ \text{AX}\end{array}\right\} (c)$$

Therefore let there be taken away on both Sides the Square of XA; there will remain

Rectangle FIA or FL;

= AS the Square of the Line BA.

Wherefore again, the common Rectangle AO being taken away,

AL will remain equal to CS.

But AL is the Square of the Line AC, seeing by the Construction AC and AI are equal. And CS is the Rectangle ABC, forasmuch as BS is equal to AB. Therefore the Rectangle ABC is equal to the Square of AC. Therefore we have cut the Line AB, as it was required.

Scholium.

THE Ten first Propositions of this Book are true also in Numbers: But this Eleventh cannot be exemplify'd in Numbers; for no Number can be so divided, that the Product of the whole multiplied by one Part shall be equal

equal to the Square of the other. The Force of this Section of a Line is wonderful, for which see *Prop.* 30. *lib.* 6.

PROP. XII. Theorem.

In an obtuse-angled Triangle (A C B), the Square of the Side (A B) opposite to the obtuse Angle (C), exceeds the Squares of the other Sides (A C, C B), by the Rectangle (B C F) twice taken; which same Rectangle is comprized under (B C), one of the Sides containing the obtuse Angle, and the Line (C F) which is intercepted betwixt the Perpendicular (A F) and the obtuse Angle. Fig. 12.

The Square A B is equal to the Squares of A F, B F (*a*). (a) Per 47. l. 1.

But the Square of B F is equal to the Squares of F C, C B, with the Rectangle F C B twice taken (*b*). Therefore if you substitute these for the Square of BF; then (b) Per 4. l. 2.

the Square of A B is equal to A F Square
F C Square
C B Square
and Rectangle B C F twice.

But the Squares of A F, F C are (*c*) equal to the Square of A C. Wherefore this being substituted for them, (c) Per 47. l. 1.

A B Square, is equal to A C Square
C B Square
+ Rectang. B C F twice.

PROP. XIII. Theorem.

In any Triangle whatsoever (as A C B) the Square of the Side (A B) opposite to an acute Angle (C) is exceeded by the Squares of the other Sides, (A C, C B) by the Rectangle (B C F) twice taken; which same Rectangle is contain'd under (B C) one of the Sides comprehending the acute Angle (C); and the Line (F C) which is intercepted betwixt the perpendicular (A F) Fig. 13, 14.

let fall upon the Side (B C) from its opposite Angle (A), and the acute Angle (C).

(a) Per 4. l. 2. The Square of B C is equal to (a) the Rectan. B F C (twice, + F C Square + F B Square.

(b) Per 47. l. 1. And A C Square is equal to (b) C F Square + F A Square.

Wherefore the two together { B C Squ. A C Squ. } are equal to Rect. B F C (twice + B F Square + 2 F C Square + A F Square.

But the Rectangle B F C twice, together with the
(c) Per 3. l. 2. Square of F C twice, is (c) equal to the Rectangle B C F twice. Therefore this being substituted for them,

B C Squ. + A C Squ. are equal to the Rectang. B C F twice + B F Square + A F Square.

(d) Per 47. l. 1. But the Squares of A F, B F are equal to (d) the Square of A B. Therefore this being substituted for them,

B C Squ. + A C Squ. are equal to the Rectangle B C F twice + A B Squ.

That is, B C Square + A C Square do exceed A B Square by the Rectangle B C F twice taken.

Corollary.

Fig. 15. THE Proposition is true, altho' the Perpendicular falleth without the Triangle. And the Demonstration is almost the same.

(e) Per 12. l. 2. [*More briefly thus.* $ACq = (e)\ ABq + CBq + 2\,CBF$. *Add on both Sides* CBq, *then* $ACq + CBq$
(f) Per 3. l. 2. $= ABq + 2\,CBq + 2\,CBF = (f)\ ABq + 2\,BCF$. Q. E. D.]

Scho-

Scholium.

FRom this Proposition and the 47th of the former Book, we have the Measure of any Triangle whatsoever, whose three Sides are known, altho the Area be altogether inaccessible. For by the help of these Theorems, the Perpendicular is known, albeit the Impediments of the Place should not permit us to mark it out. But note, That the Perpendicular, multiplied by half the Side on which it falls, produceth the Area of the Triangle, as appears out of the Scholium of Proposition 41. lib. 1.

Let there be any Triangle (as A B C) having its Sides known: It is required to give the Perpendicular A F, which falls from the given Angle A upon the opposite Side C B. *Fig. 15 or 14.*

Take the Square of the Side A B opposite to the acute Angle C, out of the Sum of the Squares of A C, and B C. By the 13th, the Remainder shall be the Rectangle B C F twice taken. Divide half of the Remainder, that is, the Rectangle B C F, by the known Side B C; thence will arise the right Line C F. Take the Square of the right Line C F out of the Square of A C. The Remainder will give (*a*) the Square of A F, whose square Root will give the Perpendicular A F. *(a) Per Prob. 2. Schol. post 47. l. 1.*

This thing also may be obtained out of Proposition 12. But the 13th sufficeth, forasmuch as in every Triangle the Perpendicular let fall from some one of the Angles unto the opposite Side, falls within the Triangle.

PROP. XIV. Problem.

THE *right-lin'd Figure* (*Q X Z*) *being given, to make a Square equal to it.* *Fig. 16.*

Make (*b*) a Rectangular Parallelogram C I equal to the Rectilinear Q X Z; the Sides of which Parallelogram, if they shall be equal, you have already made the Square which was required; if they be unequal, draw forth the greater Side I A unto L, until A L shall be equal to A C. Then bisect I L in Z; from which *(b) Per 45. l. 1.*

as from a Centre, thro' I and L describe a Circle, and let C A be produced till it meets the Circumference in B. The Square of the right Line A B is equal to the given Rectangle Q X Z.

For let the right Line Z B be drawn; because I L is cut equally in Z and unequally in A; the

Rectangle I A L + Z A Squ. } are equal (*a*) to Z L Squ. that is, equal to (*b*) Z B Square; that is, equal to (*c*) Z A Square + A B Square.

(a) Per 5. l. 2.
(b) By the Construction.
(c) Per 47. l. 1.

Taking away therefore on both Sides the common Z A *q*, there remains

Rect. I A L equal to A B *q*; that is,

Because A C and A L are equal, the Rect. C I equal to A B Square, and consequently A B Square equal to the Rectilinear (*d*) Q X Z.

(d) By the Construction.

Scholium.

EUCLID's Construction of this Problem requires that the given Rectilinear be reduc'd unto a Rectangle, by *Prop.* 45. *l.* 1. Which Reduction being operose enough, the Problem perhaps will more readily be dispatch'd after this manner.

Let the given Rectilinear be resolv'd into as many Quadrangles (X Z) as it can. Then to each Quadrangle (*e*) make an equal Rectangle. If there remain, as here it happens, one Triangle (Q), to it also (*f*) make a Rectangle equal. Then to each Rectangle by this 14. l. 2. make an equal Square; and lastly, to all these Squares let one equal one be made (*g*). This will be equal to the given Rectilinear Q X Z.

(e) Per Scho. post 45. l. 1.
(f) Per Coroll. post 42. l. 1.
(g) Per Prob. 1 Schol. post 47. l. 1.

The

The Elements of EUCLID.

BOOK III.

THE fundamental Properties of the most perfect amongst plain Figures are demonstrated in this Book. The Usefulness of the Book is manifest by this one Thing alone, that it treats of a Circle, that abundant Source of admirable Things thro' the whole Mathematicks. The more famous Theorems are *16, 20, 21, 22, 31, 32, 35, 36.*

DEFINITIONS.

1. THose Circles are equal whose Diameters or Semidiameters are equal.

2. A right Line (F B) is said to touch a Circle, when it doth so meet it in the Point (B), that albeit it be produced it doth not cut it. *Fig.* 20. *l.* 3.

3. Circles are said to touch one another, when they do so meet that they do not cut each other. *Fig.* 13, 14.

4. In a Circle the right Lines (B C, F L) are said to be equi-distant from the Centre (A), when the Perpendiculars which are let fall upon them from the Centre (A O, A I) are equal. *Fig.* 18.

5. Segments or Portions of a Circle are the Parts into which the right Line (C E) which cuts the Circle, doth divide it. *Fig.* 37.

6. An Angle in a Segment is that (B Q C) which is contain'd under the right Lines, which are drawn unto one Point of the Circumference (Q) from the Ends of the Segment, (B, C). *Fig.* 33.

7. The Angle (C Q B) is said to stand upon the Circumference (B O C), as being opposite to it. *Fig.* 33.

8. A Se-

Fig. 11. 8. A Sector is that Part of a Circle which is contain'd by two Semidiameters, (as AB, AF) and an Arch (as BF or BCF) intercepted betwixt the Semidiameters.

PROPOSITION I. Problem.

Fig. 1. l. 3. *TO find the Centre of a given Circle.*

Let the right Line (BC) be drawn in the Circle at random, which bisect in Q. Thro' Q draw the Perpendicular LF, which bisect in A. A shall be the Centre.

If you deny it; let the Centre be O, which is without the right Line FL (for in FL it cannot be, forasmuch as this Line is divided every where unequally but in A): and let there be drawn BO, QO, CO. Because therefore you suppose O to be the Centre, BO, CO, must be equal; and the Triangles BOQ, COQ, must be equilateral to each other; seeing by the Construction BQ and CQ are equal, and QO is common. Therefore the Angle OQC (*a*) is equal to the Angle OQB. Therefore OQC is a right Angle (*b*), and consequently equal to LQC which is a right one by Construction, a Part to the whole. Which is absurd.

(a) Per. 8. l. 1.
(b) Per def. 14. l. 1.

Corollary.

FRom what hath been demonstrated it appears, that if the right Line (LF) cuts another right Line BC into two equal Parts and perpendicularly, the Centre is in that Line that cuts the other.

Fig. 2. The Centre of a Circle is very easily found by a Square; the top of it (Q) being applied to the Circumference: for if the right Line DE joining the Points D and E in which the Sides of the Square cut the Circumference, be bisected in A, (A) shall be the Centre. The Demonstration whereof depends on *Prop.* 31. *l.* 3.

PROP.

PROP. II. Theorem.

IF in the Circumference of a Circle there be taken two Points (B and C) the right Line which is drawn thro' them falls entirely within the Circle. Fig. 2.

Let there be taken in the Line BC any Point whatsoever, as O, and from the Center A, be drawn AB, AO, AC. Because AB, AC are equal, the Angles also B and C are *(a)* equal. Because therefore AOC is *(b)* greater than the internal one B, it shall be greater also than C. In the Triangle therefore OAC, the Side AC subtending the greater Angle AOC, is (c) greater than the Side AO subtending the lesser Angle C. Seeing therefore AC reaches no farther than from the Centre to the Circumference, AO shall not reach so far. Therefore the Point O shall fall within the Circle. The same thing may be shew'd of any other Point of the Line BC. Therefore BC falls wholly within the Circle.

(a) Per 5. l. 1.
(b) Per Corol. 1. prop. 32. l. 1.
(c) Per 19. l. 1.

The Proposition is also manifest from the very Notion of a right Line and a Circle.

Coroll. *Hence it follows, that a right Line touching a Circle, toucheth it in one single Point only. For if it touched the Circumference in two Points, it would be a right Line drawn thro' two Points of the Circle, and consequently would fall within the Circle, contrary to the Definition of a Tangent. And by the like Reasoning (in passing from Planes to Solids) it might be prov'd, that every Plane toucheth a Sphere only in one Point.*

PROP. III. Theorem.

IF in a Circle a right Line (BL) drawn thro' the Centre bisects another (CF) not drawn thro' the Centre, it will cut it perpendicularly. And if it cut it perpendicularly, it will bisect it. Fig. 3.

Part 1. From the Centre (A) let there be drawn AC, AF. The Triangles X, and Z are equilateral to each other.

other. For C O, F O are by the Hypothesis equal, and A C, A F are so, because drawn from the Centre; while A O is common to both. Therefore the Angles A O C, A O F are (a) equal. Therefore right (b) ones. Which was the first Part.

(a) Per 8, l. 1.
(b) Per def. 14. l. 1.

Part 2. Because by the Hypothesis A O C, A O F are equal Angles; A C Square shall (c) be equal to the Squares of A O, O C together; and A F Square equal to the Squares of A O, O F together. Seeing therefore the Squares of A C, A F are equal; the Squares of A O, O C together will be also equal to the Squares of A O, O F together. Wherefore taking away the common Square A O, the Squares of O C, O F remain equal. And therefore the right Lines O C, O F are equal. Which was the other Part.

(c) Per 47. l. 1.

Coroll. (1.) *Hence in every equilateral Triangle, and in that also which is only an* Isosceles, *a Line which falling from the Top of the Angle, bisects the Base, is perpendicular to it. And on the contrary, a Line which falling from the Top of the Angle is perpendicular to the Base, doth bisect it.*

(2.) *Hence it follows, that half of the Chord of every Arch, is the right Sine of half the Arch.*

PROP. IV. Theorem.

Fig. 4, 5.

IF in a Circle two right Lines (B C, F L) not drawn both of them thro' the Centre, cut each other, they cannot bisect each the other.

Fig. 5.

For if one of them L F passeth thro' the Centre, it is manifest that it shall not be bisected by B C which doth not pass thro' the Centre.

Fig. 4.

If neither of them passes thro' the Centre, from the Centre A draw A O. If now B C, F L were both bisected in O, the Angles A O C, A O L would (d) be right Angles, and consequently equal; the Whole to a Part, which is absurd.

(d) By the foregoing.

PROP. V, VI. Theorems.

Fig. 6, 7.

CIrcles cutting each other, or inwardly touching one the other, have not the same Centre.

For

For if it were otherwise, the right Lines A B, A C F, drawn from the common Centre A, would be equal; and A C would be equal to A F a Part to the whole, because they are both equal to A B. Which is absurd.

PROP. VII. Theorem.

IF in a Circle there be taken any Point besides the Centre (A), as the Point (C), and divers right Lines fall from thence unto the Circumference (as C B, C L, C O, C F); Fig. 8.

1. *(C B) which passeth thro' the Centre will be the greatest.*

2. *The remaining Part of the Diameter (C F) will be the least.*

3. *Of the rest that will be the greater, which is nearer to the greatest.*

4. *And no more than two equal Lines can be drawn from the said Point (C) which is different from the Centre, unto the Circumference.*

Part 1. Let A L be drawn from the Centre A. Because A L, A B are equal, the common Line A C being added to each; A C and A L together are equal to C B. But A L + A C are greater than L C (*a*) Therefore C B is greater than L C. In the same manner B C will be shew'd to be greater than any other. (a) Per 20. l. 1.

Part 2. From the Centre A draw A O. A O (that is A F) is less than A C, C O (*b*). Therefore taking away the common Line A C, C O remains greater than C F. In the same manner C F is prov'd to be less than C Q, or any other. (b) By the same.

Part 3. In the Triangles C O A, C L A, the Sides L A, A C, are equal to O A, A C, each to each. But the Angle L A C is greater than the Angle O A C. Therefore (*c*) the Base L C is greater than the Base O C. (c) Per 24. l. 1.

Part 4. This is manifest from what goes before. For if there could be three drawn equal, C O, C I, C Q, there would be two on the same Side equal: Which is contrary to Part 3.

Coroll.

Coroll. *By the like reasoning* Theodosius *gathered, that of the Arches of great Circles drawn upon the Surface of a Sphere, from any Point diverse from the Pole of a certain Circle, unto that Circle, the greatest is that which passeth thro' the Pole of that Circle; the least, that which is drawn unto the opposite Point; and of the rest, that is the greater which is nearest to the greatest; as also that no more than two equal Arches can be drawn from that Point unto the Circle. And in the like manner may the Reader reason of himself on some other of the Propositions of this Book; it being very easy to pass from Planes to Solids in these Argumentations.*

PROP. VIII. Theorem.

Fig. 9, 10. *If from a Point (A) taken without a Circle, there be drawn unto the Circle the right Lines (A B, AC, A F) or (A O, A Q, A R);*

1. *Of those which fall upon the concave Circumference, the greatest is (A B) which passes thro' the Centre (Z).*

2. *Of the rest, that is the greater, which is nearer to the greatest (A B).*

3. *Of those which fall without the Circle, or upon the convex Periphery, the least is (A O) which being produced would pass thro' the Centre Z.*

4. *Of the rest, that which is nearer to the least is less than that which is farther off.*

5. *No more than two equal Lines can be drawn unto the Circumference from the same Point (A), whether they fall within the Circle, or only without.*

Fig. 9. Part 1. From the Centre Z draw Z C; because Z C, Z B are equal, the common A Z being added to each, A Z + Z C are equal to A B. But A Z + Z C are (*a*) greater than A C. Therefore A B is greater than A C. In like manner A B will be shewed to be greater than any other whatsoever.

(a) Per 20. l. 1.

Part 2. Draw Z F. In the Triangles A Z C, A Z F, the Sides A Z, Z C, are equal to A Z, Z F each to each;

each; but the Angle AZC is greater than AZF, therefore the Base AC (*b*) will be greater than the Base AF. (b) *Per* 24. *l*. 1.

Part 3. Draw ZQ. The two Lines AQ, QZ are greater than AZ (*c*). Taking away therefore the Equals ZQ, ZO, there remains AQ greater than AO. In the same manner AO is prov'd less than any other. Fig. 10. (c) *Per* 20. *l*. 1.

Part 4. Draw ZR. The right Lines AQ, QZ are less than AR, RZ (*d*); therefore the Equals, ZQ, ZR being taken away, AR remains greater than AQ. (d) *Per* 21. *l*. 1.

Part 5. This is manifest from the four foregoing.

PROP. IX. Theorem.

If from some Point within a Circle (as A) more than two equal right Lines can be drawn unto the Circumference; that Point is the Centre. Fig. 11.

This is manifest from Part 4. of Proposition 7.

PROP. X. Theorem.

Circles cut each other in two Points only. Fig. 12.

For let them cut, if it may be, in more (B, C, F.) From A the Centre of the Circle LQ, let there be drawn to the Points B, C, F, the Lines AB, AC, AF: these will be equal. Because therefore from the Point A within the Circle OS, there are drawn three equal Lines AB, AC, AF unto its Circumference, A must also be the Centre (*a*) of the Circle OS. Therefore the Circles LQ, OS, which cut one another, have the same Centre. Which contradicts Proposition 5. (a) By the foregoing.

PROP. XI. Theorem.

If two Circles touch each other inwardly, a right Line drawn thro' their Centres (A and I) passes thro' the Point of Contact (B). Fig. 13.

If you deny it, let the Centres have, if it may be, that Situation that a right Line passing thro' them shall

2 fall

fall without the Contact B, cutting the Circles in O and L. Let the Centers be A and C; and join A B, C B. Because therefore C B, C O are equal, the common A C being added to each of them, A C + C B shall be equal to A O. But A C, C B are (*a*) greater than A B, that is, than A L (*b*). Therefore also A O is greater than A L, a Part than the Whole. Which is absurd.

(a) *Per* 20. *l.* 1.
(b) By the Definition of a Circle.

PROP. XII. Theorem.

Fig. 14.

IF Circles touch one another on the out-side, a right Line which joins the Centers must pass thro' the Point of Contact.

If it be denied, let the Centers be so plac'd, as for instance in A and B, that the Line passing thro' them shall not pass thro' the Contact S, but cut the Circles in O and Q. Let the Points A S, and B S, be joined. Then A S, B S together will (*c*) be greater than A B. But A S is (*d*) equal to A O, and B S equal to B Q. Therefore A O and B Q together will be greater than A B, a part than the whole. Which cannot be.

(c) *Per* 20. *l.* 1.
(d) By the Definition of a Circle.

[Coroll. *A right Line drawn from the Centre of one of the Circles thro' the Point of Contact, will pass thro' the Centre of the other.*]

PROP. XIII. Theorem.

Fig. 15, & 16.

CIrcles touch both one another, and a right Line, in a Point only.

Fig. 15.

For let two Circles touch one another inwardly in a Part of the Circumference L C, if it may be: Then a right Line drawn thro' the Centers A and B will (*e*) pass thro' the Point of Contact, as in C. Let there be drawn also A L, B L. Because therefore B L, B C are equal (for they are drawn from the Centre B unto the Circumference O L C) the common Line A B being added, A B, B L shall be equal to A C. But A C is equal to A L, for they are both drawn from the Centre A unto the Circumference L Q C. Therefore A B, B L are equal to A L, contrary to *Prop.* 20. *l.* 1.

(e) *Per* 11. *l.* 1.

I Then

Then let the two Circles touch one another on the outside, in the Arch O L, if it may be. The right Line A P joining the Centers will pass thro' the Point of Contact (*a*), as in O for Instance. Let A L, P L, be drawn. The two Sides of the Triangle A L, P L, will be equal to A O, P O, or the whole A P; contrary to *Prop.* 20. *l.* 1. Fig. 16. (a) Per 12. l. 3.

Lastly, let the right Line B F and the Circle touch each other, if it may be, in some Part (C E): Let there be drawn unto the Centre the right Lines C A, E A. The Lines C A, E A will then be equal: And therefore the Triangle C A E is an *Isosceles*. Wherefore the Angles C and E (*b*) are acute. And therefore a Perpendicular let fall unto B F from the Centre A, will fall betwixt E and C, (*c*) as for Instance in D. There will therefore both A C and A E be equal to the Perpendicular A D; which is absurd, and contrary to *Coroll.* 14. *p.* 32. and to *Prop.* 47. l. 1. (b) Per Corol. 11. Prop. 32. l. 1. (c) Per Corol. 3. Prop. 32. l. 1.

Corollary.

Circles whose Centers are in the same right Line, and which cut it in the same Point B, do touch one another in that Point only. Fig. 17.

This Proposition is manifest from the very Notion of the Lines which are compar'd together. For neither can a right Line and the curve Circumference of a Circle, or the divers Curvatures of unequal Circumferences, or two Curvatures both convex, agree as to any Part of themselves. But they would agree, if they touched one another in some entire and proper Part.

PROP. XIV. Theorem.

In a Circle equal right Lines (B C, L F) are equally distant from the Centre (A). And what Lines are equi-distant from the Centre are equal. Fig. 18.

From the Centre (A) let there be drawn (A C, A F.) Likewise (A O, A I) at right Angles to B C, F L. Thus B C, F L shall be bisected (*d*) in O and I. (d) Per 3. l. 3.

Seeing therefore the whole Lines BC, FL are supposed equal, the halves also OC, IF must be equal; and consequently the Squares of them are also equal. Seeing therefore the Squares of AC, AF are equal, and the Square of AC is equal to OCq and OAq, as also the Square of AF is equal to IFq, and IAq (*a*): It follows that the two Squares OCq, OAq are equal to the two Squares IFq, IAq. Wherefore taking away the Squares of OC, IF (which before were shew'd to be equal) the Square of OA remains equal to the Square of AI. Therefore the Perpendiculars OA, AI are equal. Therefore (*b*) BC, FL are equi-distant from the Centre. Which was the first Part. Then for the Converse of it;

(a) Per 47. l. 1.

(b) Per defin. 4. l. 3.

If the Distances OA, AI are supposed equal, then the Squares of the equal right Lines being taken away, by the same Ratiocination it will be shew'd, that the remaining Squares OCq, IFq are equal, and consequently that the right Lines OC, IF are equal; which seeing they are * halves of the right Lines BC, FL, these also must be equal. Which was the second Part.

* Per 3. l. 3.

PROP. XV. Theorem.

Fig. 19.

OF *right Lines inscribed in a Circle, the greatest is the Diameter; and of the rest, that is the greatest, which is the nearest to the Centre.*

Let there be any Line, as RS, different from the Diameter FL. From the Centre draw AR, AS. The two AR, AS, which are equal to the Diameter, are (*c*) greater than RS. Therefore, &c.

(c) Per 23. l. 1.

Then let BI be nearer to the Centre than XZ. From the Centre unto them draw the Perpendiculars AC, AQ. AQ shall be greater (*d*) than AC. Take therefore AO equal to AC, and thro' O draw RS perpendicular to AO, which (*e*) will be equal to BI; and let AR, AS, AX, AZ be join'd. Because therefore A is the Centre, the Sides AR, AS shall be equal to AX, AZ. But the Angle RAS is greater than the Angle XAZ. Therefore the Base RS, that is, BI, is greater than the Base XZ (*f*). *Q. E. D.*

(d) Per def. 4. l. 3.

(e) By the foregoing.

(f) Per 24. l. 1.

PROP.

PROP. XVI. Theorem.

A Right Line (IF) which being drawn thro' the Point (B), the Extremity of the Diameter (CB) is perpendicular thereto, falleth all of it without the Circle, and toucheth it in (B). Neither can there any right Line be drawn betwixt it and the Circle unto the Point of Contact (B), but it shall cut the Circle. Fig. 20.

Part 1. Let there be taken in the Line IBF any Point L, unto which from the Centre A draw the Line AL. Because in Triangle ABL, the Angle ABL is a right one, by the Hypothesis, ALB shall be acute (a). Therefore AL which is opposite to the greater Angle B, will be greater than AB which is opposite to the lesser Angle L (b). But AB reacheth only to the Circumference. Therefore AL shall reach beyond the Circumference; and consequently fall without the Circle. Which was the first Part.

(a) Per Coroll. 5. P. 32. l. 1.

(b) Per 19. l. 1.

Part 2. Below BF, if it may be, let RB fall wholly without the Circle. Because FBA is a right Angle by the Hypothesis, RBA will be acute, and therefore AB is not perpendicular to BR. Therefore let there be drawn from the Centre A to BR the Perpendicular AO, which (c) will fall towards R, and cut the Circle in Q. Therefore AB which is opposite to the greater Angle AOB, is greater than AO, which is opposite to the lesser, to wit, the acute Angle OBA. But AB is equal to AQ: therefore AQ also is greater than AO, a Part than the Whole.

(c) Per Coroll. 3. Prop. 32. l. 1.

Corollary.

1. HEnce it appears again, that the Contact of a right Line and a circular one, is only in one Point. Fig. 20.

2. If from Centers taken in the same right Line infinitely protracted, there be describ'd thro' B infinite Circles, as well lesser than the first BSC, as greater; Fig. 17.

they shall all touch the right Line I F in the same one Point B.

3. Circles therefore growing into an Amplitude greater than any given one, approach always, even unto Infinity, nearer and nearer to the Tangent, but are never join'd to it, otherwise than in one single Point of Contact; which thing altho' it be most evident, is yet truly admirable.

Fig. 17. 4. From these Things it is manifest, that every geometrical Line whatsoever is infinitely divisible. For let there be drawn from some Point of the Diameter unto the Tangent the right Line A Q. Infinite Circles having Centres in the right Line B A infinitely produced; touch the right Line I F by *Coroll.* 2. of this, and one another by *Coroll. p.* 13. in one and the same Point B; and consequently are no where join'd either amongst themselves, or with the right Line I F, but in the Point B only. Therefore it is necessary that they divide the right Line A Q into infinite Parts, that is, into Parts exceeding any Number assignable.

Fig. 20. 5. The Angle of Contingence or Contact L B Q, (that, to wit, which is contained under the Tangent and the Circumference) cannot be divided by any right Line.

Fig. 17. 6. Nevertheless by Circumferences touching the Line I F in the same Point, it may be divided and diminished infinitely. And in this and the third Corollary lies hid the whole Mystery of Asymptotes, that is, of a right Line approaching unto an Hyperbola, together with it self infinitely produced, unto a Distance less than any given one, yet never concurring with it.

PROP. XVII. Problem.

Fig. 26. F*Rom a given Point (B) to draw a right Line which shall touch a given Circle (O Q).*

From A the Centre of the given Circle let there be drawn unto the Point B the right Line A B, cutting the Periphery in O. From the Centre A describe thro' B another Circle B C, and from O draw O C perpendicular to A B, which may meet the other Circle in C.

Draw

Draw CA meeting the Circle OQ in I. The right Line drawn from B unto I will touch the Circle OQ.

For since the Sides BA, IA, are equal to the Sides CA, AO, and the Angle A contain'd betwixt the equal Sides in the Triangles IAB, OAC, is common to both; the Angles AOC, AIB are also (a) equal. Therefore AIB is a right Angle. For AOC is a right one by the Construction. Therefore BI (b) toucheth the Circle in I.

(a) Per 4. l. 1.
(b) Per 16. l. 3.

Scholium.

By the 31st of this Book, from the given Point O, a Line touching a given Circle (BQ) may be well drawn thus: Fig. 27.

Let the right Line joining the given Point O and the Centre A be bisected in P. Then from the Centre P thro' A and O describe a Circle, meeting the given one in B. The right Line OB will touch the Circle.

For AB being join'd, the Angle ABO in the Semicircle is a right one by *Prop.* 31. Therefore by *Prop.* 16. OB toucheth the Circle BQ.

PROP. XVIII. Theorem.

If a right Line (CL) touch a Circle, a right Line (AB) drawn from the Centre (A) unto the Point of Contact (B) is perpendicular to the Tangent. Fig. 28.

If it be denied, let some other right Line (as AF) be the Perpendicular from the Centre A. This will cut the Circle in O. Because therefore the Angle AFB is supposed to be a right one, ABF (c) must be acute. Therefore AB (that is, AO) is greater than AF (d), a Part than the Whole; which is absurd.

(c) Per Coroll. 5. p. 32. l. 1.
(d) Per 19. l. 1.

PROP. XIX. Theorem.

Fig. 29. *IF a Line (BC) touch the Circle, and from the Point of Contact (A) there be rais'd (AI) perpendicular to the Tangent, the Centre will be in that Perpendicular.*

If you deny it, let the Centre be without AI in Z; and from it let there be drawn unto the Contact the Line ZA. The Angle ZAC will be a right one *(a)*, and therefore equal to the Angle IAC, which by the Hypothesis is a right one; that is, the Part will be equal to the Whole, which is absurd.

(a) By the foregoing.

PROP. XX. Theorem.

Fig. 30, 31, 32. *THE Angle at the Centre (BAC) is double to the Angle (BFC) which is at the Circumference, when the same Arch (BC) is the Base of the Angles.*

Fig. 30. Here are three Cases. In the first Case the Sides BA, BF coincide. And then because AF, AC drawn from the Centre are equal, there will be in the Triangle Z, the Angles F and C equal *(b)*. But BAC is equal to the two Angles F and C *(c)*. Therefore BAC is double of F.

(b) Per 5. l. 1.
(c) Per 32. l. 1.

Fig. 31. In the second Case BA, CA fall within BF, CF; and then FAX being drawn, XAB by the first Case is double of XFB, and XAC double of XFC. Therefore the whole BAC is double of the whole BFC.

Fig. 32. In the third Case, BF cuts AC, and the Angle BAC is without the Triangle BFC. Here let FAL be drawn. By the first Case the whole LAC is double of the whole LFC, and LAB taken away is double of LFB taken away. Therefore the remaining Angle BAC is also double of the remaining one BFC. *Q. E. D.*

Fig. 53. [Corollary, *Hence we gather that the Sides of every Triangle are to each other as the Sines of the Angles opposite to those Sides respectively. Let EFG be any Triangle; about which let a Circle be understood to be circumscrib'd (d), and from the Centre of the Circle, let there be let down the Perpendiculars AB, AC, AD, which*

(d) Per 5. l. 4.

*which will * bisect the Subtenses. Now as E F is to* * Per 3. l. 3. *E G, so ½ E F (that is, E B) to ½ E G (that is, E D.) But E B is the Sine of the Angle † B A E, that is, of* † Per Coroll. 2. *half the Angle E A F, that is, of the whole Angle* p. 3. l. 3. *E G F * opposite to the Side E F; and E D is the Sine* * Per 20. l. 3. *of the Angle E A D, that is, of half the Angle E A G, that is, of the whole Angle E F G, which is opposite to the Side E G. Therefore E F is to E G, as the Sine of the Angle E G F, is to the Sine of the Angle E F G.* Q.E.D. *And from this one Proposition a great Part of Trigonometry is deduced. Which Thing will be worth our Observation.*

Coroll. (2.) *From the former Corollary we learn to* Fig. 86. l. 1. *measure the Distance of the Moon. For Astronomical Observations giving us the Angle of the Diurnal Parallax * B C A, we find out the Distance of the Moon by* * Coroll. 16. *the following Proportion. As the Sine of the Angle* p. 32. l. 1. *A C B is to the Sine of the Angle A B C; so is the Semidiameter of the Earth B A, unto the Moon's Distance A C.* Q. E. I.

Coroll. (3.) *From the second Corollary we learn also to* Fig. 54. *measure the Distance of the Sun. For there being given by Astronomical Observations the Angle of the menstrual Parallax, (namely, that which is made when the Moon appears precisely bisected) or the Angle Z E O, and together with this Angle the Moon's Distance Z O; we find the Distance of the Sun by this Analogy. As the Sine of the Angle Z E O, is to the Sine of the Angle E O Z; which Sine is the Radius: So is Z O, the Moon's Distance, unto Z E the Distance of the Sun.* Q. E. I.]

PROP. XXI. Theorem.

THE *Angles (B Q C, B F C) which in a Circle* Fig 33. *stand upon the same Arch (B O C); or which are in the same Segment (B Q S C) are all equal among themselves.*

Let first the Segment B Q S C be greater than a Semicircle. From the Centre A draw A B, A C. By the foregoing the Angle B A C at the Centre is double of each B Q C, B F C. Therefore they all, B Q C, B F C, are equal (*a*). *Q. E. D.*

(a) Per axio. 6.

Fig. 34. Then let the Segment BQC be equal to or less than a Semicircle. In the Triangles BQI, CFI, because the Angles vertically opposite at I are equal (*a*), the Sum of the rest, Q and R, will be equal to the Sum of the (*b*) rest, F and O. Wherefore if from these equal Sums there be taken away the Angles R and O, which by the first Part are equal, as standing upon the same Arch QF, the Angles which remain Q, F, must be equal. *Q. E. D.*

(a) Per 15. l. 1.
(b) Per Coroll. 10. p. 32. l. 1.

Coroll. *Hence we gather in Opticks that any Line BC appears to the Eye placed any where in the Circumference of the Circle, whereof the Line is a Chord, of the same Magnitude; to wit, because it appears every where under an equal Angle BQC.*

[Schol. *If of two equal Angles standing upon the same Arch, one of them be at the Circumference, the other also will be at the Circumference.*

Fig. 33, 34. *If it be denied, BQC shall either be equal to the Angle BIC on this Side the Circumference QF, or to the Angle BEC, which is beyond the said Circumference. But the Angle BIC is* (c) *greater, and the Angle BEC* (c) *is less than the Angle BQC. Therefore, &c.*]

(c) Per Coroll. 1. p. 32. l. 1.

PROP. XXII. Theorem.

Fig. 35. *IN any Quadrilateral inscribed in a Circle (ABCF) the opposite Angles make two right ones.*

Let BF, CA be drawn. The Angle ABC with the (*d*) two O and X make two right Angles. But O is equal to I (*e*), because it stands upon the same Arch BC: And again X is equal to Z, because it stands upon the same Arch AB. Therefore ABC taken together with the two Angles I and Z, that is, with the whole opposite Angle AFC, makes two right Angles. *Q. E. D.*

(d) Per 32. l. 1.
(e) Per 21. l. 3.

[Corollary, (1.) *Hence if one Side of a Quadrilateral described in a Circle be produc'd, the external Angle will be equal to the opposite Angle of the Quadrilateral; for the Internal added to either of them makes two right Angles.*

(2.) *Likewise a Circle cannot be describ'd about a Rhombus, because its opposite Angles either fall short of, or exceed two right Angles.*

(3) Like-

(3.) Likewise if in any Quadrilateral ABCF the opposite Angles F and B are equal to two right ones, a Circle may be described about it. For (a) *a Circle will pass thro' any three Angles C, F, A, and this so that the * fourth be equal to B; which cannot be, unless it doth indeed pass thro' the Point B†. Therefore it doth pass thro' it.]* (a) Per 5. l. 4. * Per 22. l. 3. † Per Schol. pr. 21. l. 1.

PROP. XXIII, XXIV. Theorems.

ARE not necessary; and they treat of similar Segments, which cannot rightly be defin'd without Proportions.

PROP. XXV. Problem.

TO perfect a given Arch (ABC). Fig. 36.

Let there be subtended at random the two right Lines AB, CB; which bisect in I and L. From I and L raise Perpendiculars meeting one another in O. This shall be the Centre of that Circle whereof ABC is a Portion.

For (*b*) the Centre is both in the Line IX, and in the Line LZ. Therefore it is in their common Point O. (b) Per Coroll. pr. 1. l. 3.

The Practice. From the Centre B taken in the Arch describe a Circle: and with the same Interval from other Centres in the Arch describe two other Circles, each of which cuts the former twice. Two right Lines drawn thro' the Intersections, and crossing each other in O, will give the Centre.

PROP. XXVI, XXVII. Theorems.

IN equal Circles equal right Lines (CE, FI) subtend equal Arches; and if the Arches are equal, the Subtenses are also equal. Fig. 37.

These Two Propositions are plainly Axioms, and need no Demonstration.

[Coroll.

Fig. 55. [Coroll. (1.) *If in a Circle ABCD the Arch AB be equal to the Arch DC; AD will be parallel to BC. For AC being drawn, the Angles ACB, CAD, as standing on equal Arches, will be equal. Wherefore*
* Per 27. l. 1. *AD * is parallel to BC.* Q. E. D.

Fig. 56. (2.) *The right Line EF which is drawn from the Point A, the middle Point of some Arch, and toucheth the Circle, is parallel to the right Line BC, which subtends that Arch. For from the Centre D draw unto the Point of Contact A the right Line DA, and join DB, DC. The Side DG is common, and DB is equal to DC, and the Angle BDA equal to the Angle CDA, the Arches BA, CA being supposed to be e-*
* Per 4. l. 1. *qual. Therefore the Angles DGB, DGC are equal *, and consequently are right Angles. But the internal*
† Per 18. l. 3. *Angles GAB, GAF are also right Angles †. There-*
* Per 28. l. 1. *fore BC, EF are parallel *.* Q. E. D.]

PROP. XXVIII, XXIX. Theorems.

Fig. 38. IF *in equal Circles, the Angles, whether at the Centres (BAC, FLI) or at the Circumference (BOC, FSI) be equal; the Arches also (BXC, FZI) on which they stand are equal; and if the Arches be equal, the Angles also are equal.*

These two Propositions also are plainly Axioms, and need no Demonstration.

PROP. XXX. Problem.

Fig. 39. TO *bisect a given Arch (ABC.)*

Draw AC, which bisect in O. From O draw the Perpendicular OB, meeting the Arch in B. I say the thing is done.

For let AB, BC be join'd. The Sides AO, OB are by the Construction equal to CO, OB; and the Angles at O are equal, as being right ones. Therefore the Ba-
(a) Per 4. l. 1. ses AB, CB are equal *(a)*. Therefore the Arches also
(b) Per 26. l. 1. *(b)* AB, BC are equal.

The

The Practice. From the Centers A and C describe with an equal Interval, Arches cutting each other in the Points F and I, the right Line drawn thro' these Points will bisect the Arch A B C.

PROP. XXXI. Theorem.

THE *Angle (BCF) in a Semicircle, is a right one; that in a Segment greater than a Semicircle, is less than a right one; that in a Segment less than a Semicircle, is greater than a right one.* Fig. 40.

Part 1. From the Centre A draw AC. Because AB and AC are equal, the Angles O and B are equal (*a*). For the same Cause the Angles I and F are equal. Therefore the Angle BCF is equal to B and F together. Seeing (*b*) therefore the three together make two right Angles, BCF which is half of two right Angles, is one right Angle. (a) Per 5. l. 1. (b) Per 32. l. 1.

Part 2. Let the Segment LOBC be greater than a Semicircle, and in it let there be the Angle COL, and let LB the Diameter of the Circle be drawn. The Angle COL is less than that BOL, which by Part 1. is a right one. Therefore, &c. Fig. 41.

Part 3. Let the Segment LOX be less than the Semicircle LOB, and XOL be the Angle in it. This will be greater than BOL which is a right one. Therefore, &c. Fig. 41.

Corollary. *Hence we may make a Proof of the Instrument called a Square, whether it be exactly Rectangular or not. For in what Circle soever the Top of the Square is laid upon C, or any Point of the Circumference whatsoever, if the Sides of it do pass thro' the Points of the Diameter B and F, the Angle is a right one; otherwise not.* Fig. 40.

(2.) [*If the Sides of a Square be held continually upon the Points B and F, in the mean while that the Angle is moved round, first on one Side, then on the other, the Top of the Angle C will describe a Circumference of a Circle, whose Diameter is the Line B F.*]

(3.) *Hence we learn to raise a Perpendicular at the End of a Line. Let BC be the Line, C the Point given, from whence a Perpendicular is to be rais'd.*

From

From any Point whatsoever A, as the Centre, let a Circle be described passing thro' the Point C, and cutting BC in any Point, as B. If the Diameter BF be drawn, it is manifest that the Line CF is the Perpendicular required. Q. E. F.

Fig. 57. (4.) [*Hence it is manifest, that Circles touching one another inwardly, do cut all Lines, as AD proportionably; or so, that AE the Subtense of the lesser, is to AD the Subtense of the greater Circle; as AC the Diameter of the lesser, is to AB the Diameter of the greater. For there being drawn the Subtenses EC, BD, the Triangles EAC, DAB are equiangled. For the Angle A is common, and ACE, ADB are right ones, as being Angles in a Semicircle; and therefore AEC, ABD (a) are equal. The Triangles therefore are similar, by the fourth Proposition of the Sixth Book, and AC : AB :: AE : AD.* Q. E. D.

(a) Per Coral. 9. p. 32. l. 1.

Fig. 40. (5.) *In a right-angled Triangle BCF, if the Hypotenuse BF be bisected in A, the right Line AC cuts the Triangle into two equicrural ones ACB, ACF, and so a Circle described from the Centre A thro' B must pass thro' C, the top of the right Angle.*]

PROP. XXXII. Theorem.

Fig. 42, 43. *If a right Line (CF) touch a Circle, and another (AB) which is drawn from the Point of Contact (A) cut it, the Angle made by the Tangent and the cutting Line, is equal to the Angle which is made in the alternate or opposite Segment.*

That is, the Angle CAB will be equal to the Angle L, which is made in the Segment ALB; and the Angle FAB will be equal to the Angle O, which is made in the Segment AOB. For,

Fig. 42. First, let the Line AB pass thro' the Centre. Here by *Prop.* 18. CAB is a right Angle: And by *Prop.* 31. L is also a right one. Therefore CAB and L are equal.

Fig. 43. Then let the Line AB not pass thro' the Centre. Let the Line AQ therefore be drawn thro' the Centre, and BQ be join'd. Because the Angle in the Semicircle ABQ (b) is a right one, BQA taken together

(b) Per 31. l. 3.

with BAQ will make one right Angle (*a*). But CAQ (a) *Per* 32. *l.* 1.
is alſo by *Prop.* 18. of this Book a right Angle. Therefore BQA with BAQ are equal to CAQ. The common Angle therefore BAQ being taken away, there remains BQA, which is equal to L (*b*) equal to CAB. (b) *Per* 21. *l.* 3.
Therefore L and CAB are equal: Which is the firſt Part to be proved.

Then FAB and CAB make two right Angles (*c*), (c) *Per.* 13. *l.* 1.
and in the Quadrilateral BOAL, the oppoſites L and O make two right Angles (*d*). Therefore the two FAB, (d) *Per* 22. *l.* 3.
CAB are equal to the two O and L. Therefore there being taken away on one Side CAB, on the other L, which have already been ſhew'd to be equal, there remains FAB equal to O. Which was the other Part to be proved.

PROP. XXXIII. Problem.

UPon a given Line (BC) to make a Segment of a Circle, in which the Angle ſhall be equal to any Angle given. Fig. 44.

Firſt let there be an acute Angle given ABF, from B draw BL perpendicular to AB: And at C, the Extremity of the Line BC, make BCI equal to CBL (by 23. *l.* 1.) whoſe Sides ſhall cut BL in I. From the Centre I deſcribe a Circle thro' B: This Circle will alſo paſs thro' C (foraſmuch as, becauſe of the Equality of the Angles at B and C, the Sides likewiſe CI, BI are (by 6. *l.* 1.) equal) and the Segment BQC ſhall contain an Angle equal to the given one ABF.

For becauſe AB is perpendicular to the Diameter BL, AB will touch the Circle, which BC cuts (*e*). (e) *Per* 18. *l.* 3.
Therefore the Angle in the Segment BQC is equal (*f*) (f) By the foregoing.
to the Angle ABF.

But if the Angle given be obtuſe as RBC, do as before, and COB will be the Segment required.

PROP. XXXIV. Problem.

FRom a given Circle to take away a Segment containing an Angle equal to a given one. Fig. 45.

Unto the Diameter of the Circle F A draw the Perpendicular B A L. Then (*a*) let A C be drawn, which may make the Angle B A C equal to that which is given; This Line A C shall cut off the Segment A Q C, whose Angle is equal to the given one: As is manifest from *Prop*. 32.

(a) Per 23. l. 1.

PROP. XXXV. Theorem.

Fig. 46, 47, 48.

IF in a Circle two right Lines (C L, B F) cut one another, the Rectangle (C O L) under the Segments of one, is equal to the Rectangle (B O F), under the Segments of the other. For,

If they intersect each other in A the Centre of the Circle, the thing is manifest.

Fig. 46. If one of them C L passeth thro' the Centre A, and bisects the other B F which doth not pass thro' the Centre; it (*b*) cuts it perpendicularly, and so the Square of F O is the same with the Rectangle F O B. Let A F be drawn. Because C L is bisected in A and otherwise divided in O.

(b) Per 3. l. 3.

It will be thus,

(c) Per 5. l. 2.

Rect. C O L } will be equal to A L q. (*c*).
+ A O q. }
that is, to A F q.
that is, equal to A O q. }
+ F O q. } (*d*).

(d) Per 47. l. 1.

Therefore the common Square A O being taken away, there will remain

Rect. C O L equal to F O q. that is,
to the Rect. F O B.

Fig. 47. Then if one of the right Lines C L passes thro' the Centre, and cuts the other B F unequally in O, let a right Line drawn from the Centre A cut B F into two equal Parts in I. In this Case A I B will be a right Angle (*e*). Now because C L is bisected in A, and otherwise in O, it will be thus,

(e) Per 3. l. 3.

(f) Per 5. l. 2.

Rect C O L } will be equal to A L q (*f*) that is, to
+ A O q. } A B q. that is, to

A I q.

$$\left.\begin{matrix} AIq. \\ +BIq. \end{matrix}\right\} (a).$$

But AOq is equal to OIq. + AIq. (*b*). Therefore,

(a) Per 47. l. 1.
(b) By the same.

$$\left.\begin{matrix} \text{Rect. } COL \\ +OIq. \\ +AIq. \end{matrix}\right\} \text{ equal to } \left.\begin{matrix} AIq. \\ +BIq. \end{matrix}\right\}$$

Therefore the common Square AI being taken away, there remains,

$$\left.\begin{matrix} \text{Rect. } COI \\ +OIq. \end{matrix}\right\} = BIq.$$

But BI Square is equal to the Rectangle FOB, together with OI Square: (*c*) because FB is bisected in I, and otherwise cut in O. Therefore,

(c) Per 5. l. 2.

$$\left.\begin{matrix} \text{Rect. } COL \\ +OIq. \end{matrix}\right\} \text{ are equal to Rect. } \left.\begin{matrix} FOB \\ +OIq. \end{matrix}\right\}$$

Therefore the common OIq. being taken away, there remains,

Rect. COL. = Rect. FOB.

But lastly, If neither of the Lines CL, FB passes thro' the Centre: Thro' their common Intersection let there be drawn the right Line XZ, which passes thro' the Centre. By what hath been just now demonstrated, both the Rectangle COL, and that FOB, are equal to the Rectangle ZOX. Therefore COL, FOB are equal betwixt themselves (*c*).

Fig. 48.

(d) Per Axi. 1.

[*Or the Proposition may be demonstrated more easily and universally thus: Join AC and BD. Here because of the Equality of the Angles CEA, BED as being vertically opposite* (e); *and of the Angles C and B as being upon the same Arch AD* †; *the Triangles CEA, BED are equiangled* (per Corol. 9. p. 32. l. 1.) *Therefore* * *CE : EA :: EB : ED. Therefore CE×ED is equal to EA×EB* (per 16. l. 6.) Q. E. D.]

Fig. 58.
(e) Per 15. l. 1.
† Per 21. l. 3.
* Per 4. l. 6.

PROP. XXXVI. Theorem.

Fig. 49, 50, 51.

IF from (B) a Point given without a Circle there be drawn unto the Circle two right Lines, one (BF) touching it, the other (BC) cutting it; the Rectangle (CBO) which is comprehended under the whole cutting Line (CB) and the Part (BO) which lies betwixt

the

the Point (B) and the Circle, is equal to the Square of the Tangent (B F.)

1. If the cutting Line BC passes thro' the Centre A, join AF. This, with the Line FB, will make a right

(a) Per 18. l. 3. Angle (*a*). And therefore because CO is bisected in A, and to it is added OB; it will be thus.

(b) Per 6. l. 2.

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +AOq.\end{array}\right\}\text{ will be equal to } ABq.\ (b)\text{ that is, } \left.\begin{array}{l}\text{to } AFq.\\ +FBq.\end{array}\right\}(c).$$

(c) Per 47. l. 1. Therefore the equal Squares AOq. AFq. being taken away on both Sides, there remains,

$$\text{Rect. } CBO, = BFq.$$

Fig. 50, 51. 2. But then if CB doth not pass thro' the Centre, let there be drawn AB, AF, AO, and AL, and let AL bisect OC in L. The Angle ALO is therefore a

(d) Per 3. l. 3. right one one (*d*). Likewise AFB is a right Angle (*e*).

(e) Per 18. l. 3. Now because CO is bisected in L, and to it is added OB, it will be thus,

(f) Per 6. l. 2.

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +LOq.\end{array}\right\} = LBq.\ (f)$$

Let there be added on both Sides AL Square, and then

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +LOq.\\ +ALq.\end{array}\right\}\text{ equal to }\left\{\begin{array}{l}LBq.\\ +ALq.\end{array}\right.$$

(g) Per 47. l. 1. But the Squares of LO, AL are equal (*g*) to the Square of AO, or AF; and the Squares of LB, AL

(h) By the same. are equal to the Square of AB (*h*). Therefore,

(i) By the same.

$$\left.\begin{array}{l}\text{Rect. } CBO\\ +AFq.\end{array}\right\} = ABq.\text{ that is, } (i) \left.\begin{array}{l}\text{to } BFq.\\ +AFq.\end{array}\right\}$$

Therefore the common Square, that of AF being taken away, there remains

Rect. CBO equal to the Square of BF. *Q. E. D.*

Fig. 59. [*Or more easily and universally thus: Draw AB and BC. Now because of the Equality of the Angles*

*Per 32. l. 3. *A, and DBC,* * *and for that the Angle D is common*

† Per Corol. 9. p. 32. l. 1. *to both; the Triangles BDC, ADB are equiangled* †. *And therefore (by* 4. lib. 6.*) AD : DB :: BD : DC.*

(k) Per 26. l. 1. *Wherefore the Rectangle (k) AD × DC is equal to the Rectangle DB × DB or DBq.* Q. E. D.]

Coroll.

Corollaries.

1. IF from the same Point B without the Circle, as many cutting Lines, B C, as you will, be drawn, all the Rectangles, C B O, are equal amongst themselves. For each of them is equal to the Square of the Tangent B E. Fig. 51.

2. Those right Lines, which from the same Point touch the Circle are equal. For each of their Squares is equal to the same Rectangle.

[3. *It is also clear, that from the same Point B taken without the Circle, there can only two Lines B F, B Q be drawn, which shall touch the Circle. For if a third be said to touch it, it must be equal to B F or B Q, and therefore the same with one of them.*

4. *In every right-angled Triangle B F A, the Rectangle arising from the Sum of the Hypotenuse and one Side, drawn into the Difference betwixt them, is equal to the Square of the other Side. For the Sum of the Hypotenuse B A, + A F or A C, is = B C. And their Difference is B A — A F = B A — A O = B O. And the other Side of the Triangle is B F. But the Rectangle C B O is equal to the Square of B F. Therefore, &c.*] Fig. 49.

PROP. XXXVII. Theorem.

IF the Rectangle under C B and O B be equal to the Square of B F, this must touch the Circle in F. Fig. 25.

From B let there be drawn the Tangent B Q, and the right Lines E Q, E F being drawn from the Centre E, unto the Points Q and F, let B E be joined. Because by Supposition the Square of B F is equal to the Rectangle C B O, as is also the Square of B Q, by 36. of this Book; the Squares of B Q, B F shall be equal betwixt themselves, and consequently the right Lines B Q, B F are equal. Therefore the Triangles F E B, B E Q are equilateral to each other. Therefore the Angles Q, F are equal (*a*). But Q is is a right Angle (*per* 18. *l.* 3.) Therefore F also is a right Angle. Therefore B F toucheth the Circle (*b*).

(a) *Per* 8. *l.* 1.

(b) *Per* 16. *l.* 3.

[Corollaries 1. *Hence the Angle EBF is equal to the Angle EBQ* (per. 8. l. 1.)

(2.) *If two equal right Lines BF, BQ fall from some Point B upon the convex Circumference, and BF one of them toucheth the Circle, the other BQ must touch it also. For seeing BF, BQ are equal, their Squares are also equal. But BFq is equal to CBO* (a). *Therefore BQq=CBO* (b). *Therefore BQ also toucheth the Circle* (c).

(a) By the foregoing.
(b) Per Axi. 1.
(c) By this Proposition.

Scholium [1.] *Seeing all Planes passing thro' the Centre of the Earth, in which Planes all things perpendicular to the Horizon are, do produce great and equal Circles upon the Earth's Surface, we shall here bring in some elegant Confectaries from thence, out of our Author in his Astronomy; which from the Nature of Circles may very easily be understood.*

(1.) *If in any Part, the Surface of the Earth were perfectly plain, Men could no more stand upright upon it, than upon the Side of an Hill, saving in the Point of Contact only.*

(2.) *The Head of a Traveller performs a longer Way or Course than his Feet; Likewise he that is on Horseback, and goes the same way as a Footman, measures a greater or longer Space than he that is on foot. As likewise in a Ship, the uppermost Part of the Mast runs over more Way than the lower Parts of it.*

(3.) *If any one should travel over the whole Circumference of the Earth, the Way gone over by his Head would exceed that which was gone over by his Feet, by the Difference of Circumferences; or by the Circumference of a Circle, whose Semidiameter is the Man's own Stature.*

(4.) *If a Vessel full of Water be elevated perpendicularly, the Water will continually be running over, and yet it will remain full; namely, because the Surface of the Water is continually compressed into the Surface of a greater Sphere. Yea, if a Vessel be elevated continually higher and higher, the Surface of the Water which is contain'd in it, will continually descend and come nearer unto a Plane; unto which yet it will never actually come.*

(5.) *If a Vessel full of Water be carried directly downwards, altho nothing run over, yet it will cease to*

to be full; namely, because the Surface of the Water swells continually into a Part of a lesser Sphere. From whence it follows,

(6.) That one and the same Vessel contains more Water at the Foot of a Mountain than at the Top; as likewise more in a subterraneous Cellar, than in a Chamber. To which things add,

(7.) That two Strings on which two Iron Balls hang perpendicular, [and consequently the Walls of Buildings erected perpendicularly] are not parallel one to another, but Parts of Radius's meeting together in the Centre of the Earth.

Scholium. [2.] *I think it not amiss to insert in this Place this following Problem also, which was communicated to me by a Friend, as demonstrated by me somewhat more briefly.* Fig. 60.

Thro' the two Points (B) and (C) in a given Circle (B D M) to draw the Circumference of a Circle which shall bisect the Circumference of the other given Circle.

Thro' the Centre A, and one of the given Points B, let there be drawn the infinite right Line B A M E. Unto which from the Centre let there be erected the Perpendicular A D, and let B D be drawn. Let the Line D E be made perpendicular to B D, cutting the infinite Line B A M E in the Point E. Lastly, let a Circle be drawn (a) thro' the three Points, B, C, E. I say the Thing is done. For, (a) Per 5. l. 4.

Let the Chord of the second Circle be drawn thro' either of the Intersections of the Circles, as G, and thro' A the Centre of the first Circle, to wit, G A F; Let also the Diameter of the first Circle G A F be drawn. Then in the first Circle (by Corol. 1. Prop. 8. l. 6. *and by* Prop. 17. l. 6.) $AB \times AE = ADq$, *that is, (because of the Equality of the Semi-Diameters, A D, A G, A F)* $= AG \times AF$. *And in the second Circle there will be (b)* $AB \times AE = AG \times Af$. *Therefore* $AF = Af$, *and the Points F, f, coincide, and the Arch F D G is equal to the Arch F M G.* Q. E. F. (b) Per 35. l. 3.

The Elements of EUCLID.

BOOK IV.

THIS Book, which is wholly Problematical, teacheth by what Artifice, Figures, those which are ordinate or regular especially, may be inscribed in, and circumscribed about, Circles. There is very great Use of it in building Fortifications; and from it as a Fountain have been derived those most excellent Tables of Sines, Tangents, and Secants, to the very great Benefit of the Mathematicks.

[*This Book is most useful for* Trigonometry: *For by inscribing Polygons in a Circle, we learn to frame Tables of Chords, Tangents, and Secants: By the Help of which we learn to measure the Magnitudes of Figures and Bodies. Neither without it can we duly distinguish the Aspects, as they call them, of the Stars, as the Quartile, Sextile,* &c. *they wholly depending upon the Inscriptions of Polygons in a Circle. Neither can we otherwise collect the Area (which is a certain Quadrature of a Circle) than from the Area's or Squares of innumerable Polygons inscrib'd in, and circumscrib'd about, a Circle. And in like manner we know the duplicate Proportion of Circles amongst themselves, from the duplicate Proportion of Polygons inscrib'd in, or circumscrib'd about, Circles. And as for military Architecture, it makes so much Use of Polygons inscrib'd in Circles, that more than all other Sciences it may seem to be wholly owing to this Book.*]

DEFINITIONS.

1. A Rectilinear Figure is said to be inscrib'd in a Circle, or to have a Circle circumscrib'd about it, when the Tops of all the Angles thereof are in the Circumference of the Circle.

2. A rectilinear Figure is said to be circumscrib'd about a Circle, or to have a Circle inscrib'd in it, when each one of its Sides toucheth the Circle.

3. An ordinate or regular Figure is that which is equilateral and equiangular.

PROPOSITION I. Problem.

To inscribe a right Line (A) which is not greater than the Diameter into a Circle (BD). Fig. 1. l. 4.

Take in the Circumference any Point B. From the Centre B with the Interval of the given Line A, describe an Arch, cutting the Circle in C. Draw the right Line BC. I say the thing is done.

PROP. II. Problem.

To inscribe in a Circle a Triangle having equal Angles with a given one (X). Fig. 2.

Let the Line EF touch the Circle in D. Let EDG be made (a) equal to the Angle C, and FDH equal to B; and join GH. I say the Thing is done. For (b) EDG is equal to H. H consequently is equal to the Angle C (c). And FDH is equal (d) to G; and consequently G to B. Therefore GDH (e) is equal to the Angle A. Therefore what was required is done.

(a) Per 23. l. 1.
(b) Per 32. l. 3.
(c) By the Construction.
(d) Per 32. l. 3.
(e) Per Corol. 9. p. 32. l. 1.

PROP. III. Problem.

To circumscribe about a Circle a Triangle, having equal Angles with a given one (ILK). Fig. 3.

Let the Line IK be drawn forth on both Sides, so as to make the external Angles O and N. Make at the Centre A, the Angles GAB, BAF equal to O, N respectively, which is done by 23. l. 1. Then in the Points G, F, B, let three right Lines touch the Circle, meeting together in C, E, D. The Triangle CED is

circumscrib'd about the Circle, and is equi-angled to the given one I L K. For,

In the Quadrilateral C G A B, the Angles G and B are (a) both of them right ones. Therefore the remaining ones G A B, and C taken together, do (b) make two right ones, and consequently are equal to the two together, O, I. Therefore the two G A B and O, which are equal by the Construction, being taken away, there remains C equal to I. In the same manner E will be proved equal to the Angle K. Therefore D and L are (c) likewise equal. That therefore is done which was demanded.

(a) *Per* 18. *l.* 3.
(b) *Per Theo.* 1. *Schol. pr.* 32. *l.* 1.
(c) *Per Corol.* 9. *Prop.* 32. *l.* 1.

For that the Tangents do concur is thus shew'd. The Angles O, I, and K, N are (d) equal to four right ones; and I, K are less than two right ones (e). Therefore O, N, (that is by the Construction G A B, and B A F) are greater than two right ones. Therefore G A F (f) is less than two right ones. Therefore G F falls between A and D. Therefore seeing A G D, and A F D are right Angles, D G F, and D F G are less than two right ones. Therefore C G D, and E F D (g) meet together towards D. In the same manner it may be demonstrated that the rest concur.

(d) *Per* 31. *l.* 1.
(e) *Per* 32. *l.* 1.
(f) *Per Corol.* 3. *p.* 13. *l.* 1.
(g) *Per Schol. pr.* 31. *l.* 1.

PROP. IV. Problem.

Fig. 3.

TO inscribe a Circle in a Triangle.

Bisect the two Angles C and E with the Lines C A, E A, meeting together in A. From A draw the Perpendiculars, A B, A G, A F. A Circle described from the Centre A thro' B, will pass also thro' G and F, and touch the three Sides of the Triangle.

For in the Triangles C A G, C A B, because the Angles A G C, A B C, and likewise those G C A, and B C A are equal by the Construction, and the Side A C is common, the Sides A G, A B (h) must be likewise equal. In like manner A B, A F may be shewn to be equal. Therefore the Circle describ'd from the Centre A, passeth thro' B, G, F. And because the Angles at those three Points are equal, it toucheth (i) all the Sides of the Triangle. That therefore is done which was required.

(h) *Per* 26. *l.* 1.
(i) *Per* 16. *l.* 3.

[Hence

[Hence the Sides of a Triangle being known, the Segments of them which are made from the Contacts of an inscribed Circle will be known. Let DC be 12. DE 18. CE 16. DC and CE will be 28, from which subtract 18 = DE = DG + BE, there remains 10 = CG + CB. Therefore CG or CB = 5. Consequently EB or EF = 11. Wherefore FD or DG = 7.]

PROP. V. Problem.

To describe a Circle about a Triangle, or thro' three given Points B, C, D, not lying in a right Line, to describe a Circle. Fig. 4.

Connect the given Points with two right Lines BC, CD, which bisect with the Perpendiculars EA, OA, meeting together in A. This will be the Centre of a Circle which passeth thro' B, C, D.

Let the right Lines AC, AD, AB be drawn. By the Construction the Sides DO, OA are equal to these CO, OA; and the Angles at O are right ones. Therefore AD is equal to AC (a). In the same manner AB may be prov'd equal to AC. Therefore AD, AB (b) are equal. Therefore a Circle described from the Centre A thro' B, will pass also thro' C and D. Which was the Thing required.

(a) Per 4. l. 1.
(b) Per Axi. 1.

As for the Practice, it is sufficient to describe from B, C, D three equal Circles, intersecting each other; and thro' the Intersections to draw right Lines, these meeting one another will give the Centre sought.

PROP. VI, VII. Problems.

To inscribe a Square in, and circumscribe one about a Circle. Fig. 5.

Let the Diameters BD, CE be drawn, cutting each other perpendicularly. The right Lines which join the Terms of these, inscribe a Square in a Circle.

The Demonstration is manifest from 4. l. 1. and 31. l. 3. Then let four Tangents be drawn touching the Circle in B, C, D, E, meeting together in I, F, G, H. The Figure IFGH is a Square, circumscrib'd about a Circle.

The Demonstration is manifest from 18. l. 3. with Coroll. 2. Prop. 36. l. 3. and 28, and 34. l. 1.

Scholium.

Fig. 5. A Square describ'd about a Circle is double to that inscrib'd. For because the Angle BCD in the Semicircle (a) is a right one, the Square of BD (that is FI Square) shall be (b) equal to BCq + CDq. and therefore double to the Square of CD, *i. e.* to CDEB.

(a) Per 31. l. 13. (b) Per 47. l. 1.

PROP. VIII, IX. Problems.

Fig. 6. *To inscribe a Circle in, and circumscribe one about a Square, (as BCFE.)*

Let there be drawn the Diameters of the Square, cutting each other in O. From the Centre O describe a Circle thro' B; this will also pass thro' E, F, C.

Then from the Centre O draw OD perpendicular to BC; a Circle describ'd from the Centre O thro' D, will touch all the Sides of the Square.

Part I. Because by the Hypothesis the Lines CB, EB are equal; the Angles BCE, BEC will be equal (c). But CBE is a right Angle by the Hypothesis. BCE therefore and BEC are half right ones (d). In the same manner CBF will be shew'd to be an half right Angle, as likewise the rest of the Angles; and so they are equal amongst themselves. Therefore in the Triangle BAC, seeing there are two equal Angles CBO, BCO, the right Lines OB and OC (e) are equal. In like manner the right Lines OB, OE, OF may be shew'd to be equal. Therefore a Circle described from the Centre O thro' B, passes thro' E, F, C.

(c) Per 5. l. 1. (d) Per Corol. 11. Prop. 32. l. 1. (e) Per 6. l. 1.

Part 2. From O let there be also drawn the Perpendiculars OG, OH, OI. Because in the Triangles GBO,

DBO,

DBO, the Angles at D and G, as likewise those at B are equal, and the Side OB is common, the Sides OD, OG must be equal *(a)*. In the same manner OG, OH, OI may be shew'd to be equal. Therefore a Circle describ'd from the Centre O, which passeth thro' D, will also pass thro' G, H, I, and touch all the Sides of the Square *(b)*, because the Angles at D, G, H, I are right ones. Therefore we have done what was required. (a) Per 26. l. 1. (b) Per 16. l. 3.

PROP. X. Problem.

To make an Isosceles Triangle BAC, in which the Angle at the Base (ABC, or ACB) shall be double to that which is at the Top (A). Fig. 7.

Let any right Line, hat you will, as AB, be taken, which so cut in D *(c)* that the Rectangle ABD shall be equal to AD Square. Then from the Centre A thro' B describe a Circle; in which inscribe *(d)* BC equal to AD, and join AC. BAC shall be the Triangle sought. (c) Per 11. l. 2. (d) Per 1. l. 4.

For let the right Line DC be drawn, and thro' A, D, C describe *(e)* a Circle. Because the Rectangle ABD is equal to the Square AD, (that is, BC,) it is manifest, that BC *(f)* toucheth that Circle DC which CD cuts. Therefore the Angle BCD *(g)* is equal to the Angle A in the opposite Segment; and so the common Angle DCA being added, BCA must be equal to A+DCA. But because the Sides AB, AC are equal, ABC *(h)* is equal to the Angle ACB. Therefore the Angle ABC is also equal to A+DCA. But the external Angle also BDC is equal to the two internal ones *(i)* A+DCA. Therefore ABC, and BDC are equal. Therefore the Line DC is *(k)* equal to BC, (that is, by the Construction to DA). Therefore the Angles A and DCA *(l)* are equal. Wherefore the Angle ABC, which hath been shew'd equal to those two, shall be double to one, A. That is done therefore which was required. (e) Per 5. l. 4. (f) Per 37. l. 3. (g) Per 32. l. 3. (h) Per 5. l. 1. (i) Per 32. l. 1. (k) Per 6. l. 1. (l) Per 5. l. 1.

Corollary.

EAch of the Angles at the Base B and C in the *Isosceles* now framed, is two fifths of two right ones, or four fifths of one right one, and the remaining one A is one fifth of two right ones, or two fifths of one right one: As is manifest out of this Proposition taken together with 32. l. 1.

PROP. XI. Problem.

Fig. 7. 8. TO *inscribe a regular Pentagon in a Circle.*

(a) By the foregoing. Let there be described (*a*) the Triangle B A C, having the Angles at the Base double to that at the Top. Inscribe a Triangle C A D equiangled to this, in a Circle (*b*). Bisect the Angles at the Base A C D, A D C, with the right Lines C E, D B, cutting the Circle in E and B. The Points A, B, C, D, E, join'd by right Lines, will give an ordinate Pentagon inscrib'd in a Circle.

(b) Per 2. l. 4.

For from the Construction it appears that the Angles I, N, Q, S, O are equal. Wherefore the Arches subtended to them A E, E D, C D, C B, B A are also (*c*) equal. Therefore the right Lines subtended to those Arches shall also (*d*) be equal. The Pentagon therefore is equilateral. But it is also (*e*) equiangular, because its Angles B A E, A E D, &c. stand on equal Arches B C D E, A B C D, &c. That therefore is done which was required.

(c) Per 28. l. 3.
(d) Per 27. l. 3.
(e) Per 29. l. 3.

Corollary.

Fig. 8. THE Angle of a regular Pentagon makes six fifths of one right Angle, or three fifths of two. For the three Angles at A, seeing they are equal, as standing upon equal Arches, B C, C D, D E, and the middlemost of them, by the Corollary foregoing, is two fifths of one right Angle; the three together, that is, the Angle of the Pentagon it self, must make six fifths of one right one.

[Scho-

[Scholium. *This holds universally, that Figures of an odd Number of Sides are inscrib'd in a Circle, by means of an* Isosceles *Triangle, whose equal Angles at the Base are multiple of those at the Top. But Figures of an even Number of Sides are inscrib'd by the means of* Isosceles *Triangles, whose Angles at the Base are each of them multiple sesquialteral of that which is at the Top.* Fig. 8.

As in the Isosceles *ACD, if the Angle C or D be threefold of A, the Side CD will be the Side of an Heptagon; if fourfold, it will be the Side of an Enneagon,* &c. *But if C or D shall be* $1\frac{1}{2}$ *of A, CD will be the Side of a Square; and if C shall be* $2\frac{1}{2}$ *of the Angle A, CD will subtend a sixth Part of the Circumference: In like manner, if C or D shall be* $3\frac{1}{2}$ *of the Angle A, CD shall be the Side of an Octagon,* &c.]

Scholium.

EUclid's Inscription of a Pentagon, is ingenious, but that of *Ptolemy*, which he delivers in the first Book of his *Almagest*, is much more expeditious: And it is this.

Let the Diameters E D, B F, be drawn, cutting one another perpendicularly in A. Bisect the Radius A D in C. From the Centre C thro' B describe an Arch, meeting the Diameter E D in G. The right Line G B is the Side of a Pentagon, and A G of a Decagon. Fig. 12.

The Demonstration cannot be given here, for it depends upon the 13th Book of *Euclid*. See it in *Clavius*, in his Scholium, after *Prop.* 10. *l.* 13.

Problem.

UPON a given right Line (A B) to describe a regular Pentagon. Fig. 9.

Cut A B so in C (a) that the Rectangle A B C may be equal to the Square of A D. From A B protracted on both Sides take away A D, B E, equal to the greater Segment A C. From the Centers A and D with the Interval A B describe two Arches, cutting each other in F. Likewise from the Centers B and E describe, with the (a) Per 11. l. 2.

same

ſame interval, two Arches cutting each other in G. And again, from the Centers G and F, with the ſame Interval, deſcribe two others, cutting each other in I. The Points A, F, I, G, B, being join'd, will give a regular Pentagon upon the right Line A B.

That it is equilateral, 'is manifeſt from the Conſtruction; that it is equi-angled, will be thus demonſtrated. Let D F be drawn. It is manifeſt by the Conſtruction, that A D F is an *Iſoſceles*. And the Baſe A D is the greater Segment of the Side D F, ſo divided, that the Rectangle of the whole and the leſſer Side, is equal to the Square of the greater. (For D F is equal to A B, and A D equal to A C.) Therefore the Angle D A F is two fifths of two right ones; by *Coroll. Prop.* 10. *l.* 4. Therefore the remaining Angle F A B is three fifths of two right ones, or ſix fifths of one right one (*a*); and therefore is an Angle of a regular Pentagon (*b*). In the ſame manner may it be ſhewn, that the Angle G B A is three fifths of two right ones, and ſo equal to F A B. From whence it is neceſſary, that the reſt, F, G, I, ſhould be equal to theſe, as appears from their being equilateral to theſe, if the right Line F G be conceiv'd to be ſubtended.

(a) *Per* 13. *l.* 1. (b) *Per Corol. p.* 11. *l.* 4.

PROP. XII. Problem.

Fig. 10.

TO *circumſcribe an ordinate Pentagon about a Circle.*

Let there, by the foregoing, be inſcrib'd the regular Pentagon G H I K M, and let there be drawn Tangents in the Points G, H, I, K, M, which may concur in B, C, D, E, F. I ſay the thing it done.

For from the Center draw the right Lines, A G, A B, A H, A C, A I. Here becauſe from the ſame Point B, B G, and B H touch the Circle, they (*c*) are equal. Therefore the Triangles G A B, B A H are equilateral to each other. Therefore (*d*) the Angles O P, as likewiſe thoſe Q, S, are equal. And therefore the whole Angle B is double to P, and the whole G A H double to S. For the ſame Reaſon the Angles C and H A I are double to T and N reſpectively. But G A H and H A I are equal (*e*), becauſe they ſtand upon equal Arches by Conſtruction, G H, H I. Therefore their halves S and N are

(c) *Per Corol.* 2. *p.* 36. *l.* 3. (d) *Per* 8. *l.* 1. (e) *Per* 29. *l.* 3.

N are alſo equal. Becauſe therefore in the Triangles BAH, HAC, the two Angles S and N are equal, and thoſe at H are both right Angles (*a*), and likewiſe the Side AH is common; therefore the Sides (*b*) BH, CH, as likewiſe the Angles P, T, are equal. In the ſame manner I might ſhew BG, FG to be equal. Therefore BF, CB which are double to the equals BG, BH, are alſo equal. In the ſame manner it may be ſhew'd that the reſt of the Sides of the circumſcribed Pentagon are equal. It is therefore equilateral; but it is alſo equiangled; for ſeeing it hath been ſhew'd that the Angles B and C are each of them double to the Equals P, and T, they muſt alſo be equal betwixt themſelves. And in the ſame manner of the reſt. We have therefore deſcribed a regular Pentagon about a Circle: Which was the thing to be done.

(a) Per 18. l. 3.
(b) Per 26. l. 1.

In the ſame way any ordinate Figure whatſoever is deſcrib'd about a Circle, that is, if a like Figure be firſt inſcrib'd in the Circle.

PROP. XIII, XIV. Problems.

To inſcribe a Circle in a regular Pentagon, and circumſcribe one about it.

Biſect the two Angles of the Pentagon B, C, with the right Lines BN, CS, cutting each other in A. From A draw the Perpendicular AL. Fig. 11.

A Circle deſcrib'd from the Point A, with the Interval AL, touches all the Sides of the Pentagon; and a Circle deſcrib'd from the ſame Point A, with the Interval AB, paſſes alſo thro' the Points F, E, D, C.

Part I. In the Triangles DCA, BCA, becauſe the Sides DC, CA, are equal to BC, CA, by the Hypotheſis, and the Angles P and O are equal by the Conſtruction, thoſe alſo G and I will be equal by 4. *l.* 1. Now the whole alſo B and D are equal by the Hypotheſis. Wherefore ſeeing the Angle G is half of B by the Conſtruction, I will alſo be half of D. Therefore D is biſected by the right Line DM. For the ſame Cauſe the reſt of the Angles of the Pentagon E, F, are biſected, and conſequently all the half Angles are equal betwixt themſelves. Now let the Perpendiculars be

drawn,

drawn, AM, AS, AN, AR. Since then in the Triangles LBA, MBA, the Angles G and BLA are equal to the Angles Q and BMA, by the Construction, and the Side BA is common, AL and AM must be also equal (*a*). In like manner I might shew that the rest of the Perpendiculars, AM, AN, AS, AR, are equal. A Circle therefore from the Centre A, passing thro' L, will likewise pass thro' M, S, N, R; and because the Angles at L, M, S, N, R, are right ones by the Construction, * it will touch the five Sides of the Pentagon. Which was the first Part.

(a) Per 26. l. 1.

* Per 16. l. 3.

Part 2. In the Triangle CAB because the Angles O and G have already been shewn to be equal, the Sides also AC, AB must be equal (*b*), and in the same manner, AB, AF, AE, AD, may be prov'd equal; and therefore a Circle from the Centre A passing thro' B must pass also thro' C, D, E, F. Therefore we have both inscrib'd a Circle in a Pentagon, and circumscrib'd one about a Pentagon. *Q. E. D.*

(b) Per 6. l. 1.

[*In the same way, in any regular Figure whatsoever, a Circle may be inscrib'd, and circumscrib'd about it.*]

PROP. XV. Problem.

Fig. 13.

IN a given Circle to describe a regular Hexagon.

Let the Diameter FAB be drawn. From the Centre B, thro' A, describe a Circle, cutting the given one in C and D. Likewise from the Centre F, thro' A, a Circle, cutting the given one in E and G. The six Points, B, C, E, F, G, D, connected by right Lines will give the Hexagon required.

From the Centre A let fall the right Lines AE, AC, AG, AD. It is manifest that the Triangles H, I, M, L, are equilateral, both in themselves, and with one another (*c*). Then because the Angles CAB, EAF, each of them make one third of two right Angles (*per Corol.* 12. *p.* 32. *l.* 1.) and therefore do make both together two thirds of two right Angles; it remains (*d*) that EAC is one third of two right Angles; therefore the Angles EAC, CAB are equal. But the Sides also EA, AC, are equal to the Sides BA, AC. Therefore the Base EC

(c) Per 1. l. 1.

(d) Per Corol. 1. p. 13. l. 1.

(*per* 4. *l.* 1.) is equal to the Base BC, that is, to the Radius AC by the Construction. Wherefore the Triangle N is also equilateral. And in the same manner the Triangle K may be shewn to be so. Because therefore all the six Triangles, H, I, K, L, M, N, are equilateral; it is manifest that all the Sides, CB, BD, DG, GF, FE, EC, are equal one to another, and to the Radius AC. The Hexagon is therefore equilateral. But it is also equiangular, seeing each one of its Angles E, C, B, D, G, F, consists of two Angles of an equilateral Triangle. Therefore we have inscribed a regular Hexagon in the Circle.

Corollaries.

1. THE Side of an Hexagon inscrib'd in a Circle, [or a Chord of 60 Degrees] is equal to the Radius [and consequently the Sine of 30 Degrees is equal to half the Radius (*per Corol.* 2. *p.* 3. *l.* 3.)]

2. An Angle of a regular Hexagon is four thirds of one right Angle; as consisting of two Angles of an equilateral Triangle, each of which makes two thirds of a right Angle.

3. If there be drawn the Diameter PS, perpendicular to the other FB; and with the Interval of the Radius PA, from the Centre P, and S, there be made Sections in O and Q, in R and T, and in like manner from the Centers F and B, make the Sections in G and E, in D and C; the Points, P, E, O, F, R, G, S, D, T, B, Q, C, connected with right Lines, will give a Figure of 12 Sides, inscrib'd in a Circle with one Aperture of the Compasses. Which Thing is of great Service in Dialing. *Fig.* 14.

4. From what has been demonstrated we may easily describe an equilateral Triangle in a Circle. The Diameter FB being drawn, from the Centre B thro' A describe the Arch CAD. The Points C, F, D, connected with right Lines, will give the Triangle sought. *Fig.* 14.

5. The Side CXD of an equilateral Triangle, cuts off from the Diameter BF perpendicular to it, a fourth Part thereof BX. For the Angles ACX, BCX, standing upon equal Arches GD, DB are equal (*per* 29. *l.* 3.) and the Sides AC, CX, are equal to the Sides BC,

CX.

(a) Per 4. l. 1. CX. Therefore AX, BX are equal (*a*). Therefore BX is the fourth Part of the Diameter BF.

Scholium 1. *Problem*.

Fig. 13. * Per 1. l. 1. YOU may raise a regular Hexagon upon a right Line BC thus. Make an * equilateral Triangle, CAB upon the given Line CB. From the Centre A thro' B and C describe a Circle. This will contain an Hexagon upon the given right Line CB. The Thing is manifest from the Proposition, and *Coroll.* 1.

Theorem.

THE Square of a Side of an equilateral Triangle is triple to the Square of the Semidiameter of a Circle in which it is inscrib'd, and is to the Square of the whole Diameter, as 3 to 4.

Fig. 14. Let there be drawn the Semidiameter AD. The Square of FD is equal to FAq + DAq + the Rectangle FAX twice taken (*per* 12. *l.* 2.) But the Rectangle FAX twice taken is equal to the Square of the Semidiameter FA or DA: (for because AX, XB (*b*) are equal, the Rectangle FAX twice taken, is equal to the two Rectangles which are under FA, AX, and under FA and XB, that is, equal to the Rectangle FAB (*c*); that is, equal to FAq.) Therefore FDq is triple to FAq or DAq the Square of the Semidiameter.

(b) Per Coroll. 5. foregoing.

(c) Per 1. l. 2.

Now because the Square of the whole Diameter is quadruple of the Square of FA the Semidiameter (*d*), it is manifest that the Square of FD is to the Square of the Diameter, as 3 to 4.

(d) Per Coroll. 3. prop. 4. l. 2.

Hence it follows that a Side of an equilateral Triangle is to the Diameter, as the square Root of 3 is to 2, the square Root of 4; and therefore that those Lines are incommensurable.

PROP.

PROP. XVI. Problem.

TO *inscribe a regular Quindecagon in a Circle.* *Fig.* 15.

Inscribe in the Circle AC the Side of a Pentagon (*a*), (a) *Per* 11. and AD the Side of an equilateral Triangle, (*per Corol.* 4. 4. *p.* 15. *l.* 4.) bisect the Arch CD in E. CE is the Side of the Quindecagon, or fifteen-angled Figure sought.

For if the whole Circumference be suppos'd to be 15, the Arch AC will be 3, and the Arch AD 5, and therefore the Arch CD 2, and consequently CE 1.

Corollary.

BY this Method innumerable regular Figures may be inscrib'd in a Circle. For if AC, AD, the Sides of two regular Figures be inscrib'd in a Circle, the Difference of the Arches CD will contain so many Sides of a new regular Figure, as are the Units whereby the Denominators of the former differ one from another. But the Denominator of the new Figure is had, if the Denominators of the former be multiplied one by the other. *Fig.* 15.

As if AD be the Side of a Square, and AC of a Decagon, the Difference of the Denominators is 6. Therefore the Arch CD contains 6 Sides of a new Figure. But the new Figure is of 40 Sides. For the Denominators 4 and 10 multiplied one by the other make 40.

Scholium.

THere hath not yet been found out the Art by which regular Figures of 7, 9, 11, 13, 17, &c. Sides may be inscribed in a Circle, by a Pair of Compasses and a Rule only; forasmuch as that Inscription of Figures depends upon the Division of the Circumference into any given Parts, which thing is lacking: But if the Circumference of a Circle be divided into 360 Parts, you may in a mechanical way inscribe any regular Figures whatsoever, in it, after this manner.

H Pro-

Problem 1.

Fig. 15. DIvide 360 Degrees (that is, the whole Circumference) by the Denominator of the Polygon to be inscrib'd (*e.g.* a Nonangle). Make at the Centre the Angle A G K, of so many Degrees as are the Units of the Quotient in the said Division. A K shall be the Side of the nine-angled Figure, which is required to be inscrib'd in the Circle.

Problem 2.

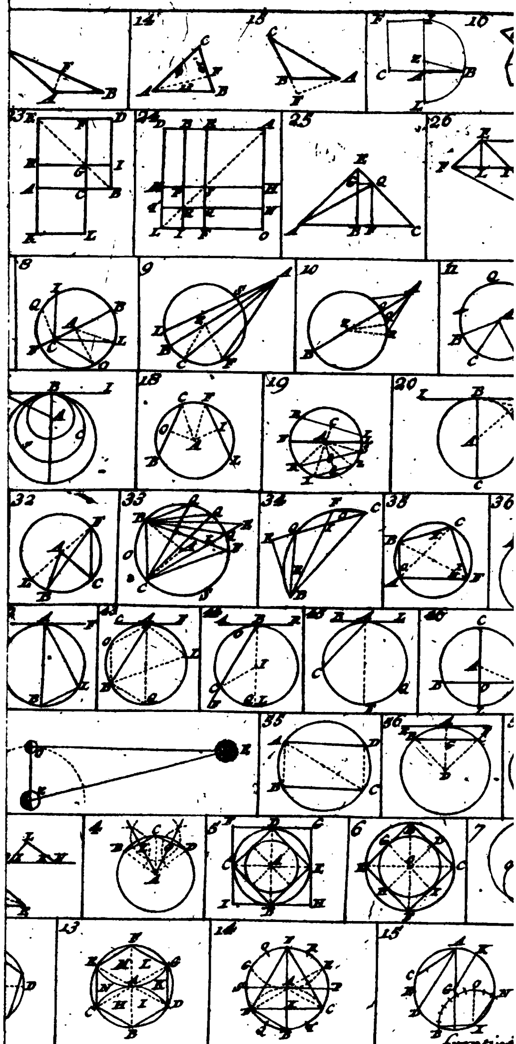
Fig. 15. BUT upon a given right Line you may describe any regular Figure whatsoever by the Help of the following Table.

A right Angle is to the Angle of the Figure,

			Difference.
In a Pentagon as	5	to	6—1
In an Hexagon as	3	to	4—1
In an Heptagon as	7	to	10—3
In an Octagon as	2	to	3—1
In a Nonagon as	9	to	14—5
In a Decagon as	5	to	8—3
In an Undecagon as	11	to	18—7
In a Duodecagon as	3	to	5—2

Fig. 15. Let a regular Heptagon be to be described upon the given right Line X B. From the Centre X, with the Interval X B, describe a Circle, from which cut off the Quadrant B O. See in the Table what is the Proportion of a right Angle to the Angle of a Heptagon: You will find it to be as 7 to 10, and the Difference is 3. Divide the Quadrant therefore into seven equal Arches, so many of which add to it from O to N as the Difference hath Units. Thro' the three Points B, X, N, describe (*per* 5. *l.* 4.) a Circle. This contains an Heptagon on the given right Line X B.

The Table was made by means of Theorem 2. in the *Schol.* upon *p.* 32. *l.* 1. by which is found the Number of right Angles, which the Angles of any right-lin'd Figure make; which Number being divided by the Deno-



es, which the Angles of any right-lin'd Fi-
which Number being divided by the Deno-

2

minator

minator of the Figure, gives the Denominator of the Proportion of the Angle of the Figure to a right one.

Now because hitherto many Things have been propounded concerning regular Figures, let the following famous Theorem of *Proclus* close this Book.

Theorem.

ONly three regular Figures, to wit, 6 equilateral Triangles, 4 Squares, and 3 Hexagons, can fill a Space; that is, can constitute one continu'd Superficies. Which is thus demonstrated. That some regular Figure often repeated should be able to fill a Space; it is required that the Angles of many Figures of that kind being disposed about one Point, should make just four right ones; for just so many right Angles may be placed about one Point, as appears from *Corol.* 3. *Prop.* 13. *l.* 1. As for Example, that equilateral Triangles should fill a Space, it is requir'd that so many Angles of such Triangles N, M, L, K, I, H, being dispos'd about the Point A, should make just four right ones. But six Angles of an equilateral Triangle do make four right ones; (for one makes two thirds of one right one *, and therefore six of them make 12 thirds of one right one, that is, 4 right ones:) Likewise the four Angles of a Square make four right ones, as is manifest; likewise three Angles of an Hexagon; for one maketh four thirds of one right Angle; (*per Corol.* 2. *p.* 15. *l.* 4.) and therefore three of them do make twelve thirds of one right Angle, that is, four right ones. Therefore, *&c.*

Fig. 13.

* *Coroll.* 12. *p.* 32. *l.* 1.

That no other Figure besides these can do this, will manifestly appear, if its Angle being found as above, you shall multiply the same by any Number whatsoever; for the Angles will always either fall short of, or exceed four right ones.

The Elements of EUCLID.

BOOK V.

THIS fifth Book of Elements is altogether necessary for demonstrating the Propositions of the Sixth Book. The Doctrine which it containeth is almost in continual Use. The Way of Reasoning from Geometrical Proportion is most subtle, solid and brief. This Method of Reasoning, as a kind of Mathematical Logick, Geometry, Arithmetick, Musick, Astronomy, Staticks, and all the other Parts of the Mathematicks, make especial Use of: Forasmuch as they almost wholly depend upon Proportions connected together one with another; and are wont to borrow their Ways of Reasoning concerning Proportionals from this Fifth Book. Practical Geometry, which consists in the measuring of Lines, Figures, and Solids, is for the most part derived from the Doctrine of Proportions. There is not a Rule in Arithmetick but what may be demonstrated from the Propositions of this fifth Book, without the help of the 7th, 8th, and 9th Books, which treat professedly of Numbers. We may fitly call the Musick of the Antients, Geometrical Proportions apply'd to tuneful Sounds; which same Thing you may well nigh say concerning Staticks, which are conversant about the Weights of Bodies. To comprehend the whole Matter in few Words; If you take away the Doctrine of Proportion from the Mathematicks, you will leave almost nothing which is excellent or greatly to be accounted of.

Scholium.

There is no Mathematician, who is ignorant of how great Importance in Geometry the Knowledge of Proportions is; for it is the very Marrow, as it were, of the

the Mathematical Sciences: and the various Ways of Reasoning concerning Proportionals, are both most useful, and most certain; neither can we without them move one Step.

But then I reckon that this Doctrine is congenite in Men's Minds with common Reason it self; and that the various Ways of Reasoning concerning Proportionals, which Euclid, *by much winding and going about, delivers in this whole Book, do not so much need Demonstration, properly so call'd, as Illustration and Examples. And I am altogether of Opinion that those who take in Hand to deliver this most easy Doctrine by a long Circuit of Propositions, do involve a Thing in it self most clear, in a certain Cloud and render it far more difficult. The Sum of the Matter I will open in a few Words. It is a thing easily known, that four Quantities are then proportional, or that the Analogies are then alike, when the first Quantity contains the second, as often as the third contains the fourth; or when the first is as often contain'd by the second, as the third is by the fourth. So* 16 : 8 : : 4 : 2. *And* 3 : 9 : : 4 : 12. *are like or the same Proportions; because in the former Example the Consequents* 8 *and* 2 *are contain'd twice in their respective Antecedents; and so the Proportion of the Antecedents to the Consequents is double. And in the other Example the Proportions are also alike; because the Consequents* 9 *and* 12 *do contain their respective Antecedents three times; and so the Proportion of the Antecedents to the Consequents is subtriple. (Nor is there any Proportion of commensurable Quantities which may not be express'd by certain* Numbers; *nor indeed of Incommensurables, which may not be expressed by* Numbers *infinitely approaching nearer and nearer unto the true one.) Furthermore from what hath been said it appears, that like Proportions, whatsoever they are, may be express'd not only by divers* Numbers, *but also by the same. Thus* 2 *to* 1 *designs as well the Proportion of* 16 *to* 8, *as of* 4 *to* 2. 1 *to* 3 *no less expresseth that of* 4 *to* 12, *than that of* 3 *to* 9, *as is most manifest. Supposing therefore four Quantities to be proportional,* $A : B : : a : b$; *it is enquir'd in this Book, after how many like Manners the Terms of these like Proportions may be changed, and ordered amongst themselves; so that the emerging Proportion on both Sides may be still alike? And it may be*

 answer'd,

answer'd, that it may be done after all the ways and manners possible; for seeing the Proportion of A to B, and that of a to b are alike, both of them may be express'd by the same *Numbers after this manner*, $A : B :: 9 : 3$, *and* $a : b :: 9 : 3$. *And consequently all the Proportions emerging on both Sides, either by* Alternating *the Terms, or by* Inverting *them, or by* Compounding, *or* Dividing, *or* Converting, *or* Mixing *them, may be express'd by the* very same *Numbers; and consequently the same Proportion will always be kept on both Sides. As for Example sake.* $A+B : B :: a+b : b$, *because* $9+3 : 3$, *expresseth the same Proportion; which is* Composition. *The same is to be said of all the ways of changing the Terms. Therefore let Beginners observe this one Thing, that Proportions, which are on both Sides the same, be ever changed and ordered in the very same manner. And then there will be no Room to question, whether the Proportions which arise on both Sides be alike or no. It is indeed a Thing to be wonder'd at, that no one of those who have hitherto compiled Elements of Geometry, have made use of this most easy Method of stating the Equality of Proportions, for the Illustrating of this Fifth Book about the Doctrine of Proportions. Take therefore the primary Ways which Geometry makes use of, in reasoning concerning like Proportions, as they are digested into this short Table.*

Let it be	$A : B :: a : b :: 9 : 3$
Then it will be by	
Alternating	$A : a :: B : b :: 9 : 9 :: 3 : 3$
Inverting	$B : A :: b : a :: 3 : 9$
Compounding	$A+B : B :: a+b : b :: 9+3\ (12) : 3$
Dividing	$A-B : B :: a-b : b :: 9-3\ (6) : 3$
Converting	$A : A+B :: a : a+b :: 9 : 9+3\ (12)$
or	$A : A-B :: a : a-b :: 9 : 9-3\ (6)$
Mixing	$A+B :: A-B :: a+b : a-b :: 9+3 : 9-3$
Ex æquo	$A : B :: a\ b$, & $B : C :: b : c$, *then* $A : C :: a ; c$,
	$9 : 3 :: 9 : 3$, $3 : 1 :: 3 : 1$, *then* $9 : 1 :: 9 : 1$.
Ex æquo perturbatè.	$A : B :: a : b$, & $B : C :: r : a$, *then* $A : C :: r : b$.
Or thus,	$8 : 3 :: 8 : 3$, & $3 : 12 :: 2 : 8$, *then* $8 : 12 :: 2 : 3$.
	$8 : 3 :: 16 : 6$, & $3 : 2 :: 24 : 16$, *then* $8 : 2 :: 24 : 6$.
	$a : b :: e+a : a+b$ & $b ; e :: a+b : a+e$, *then* $a\ e ::: a+b : e+b$.

He-

He therefore who is expert in these Ways of Reasoning concerning Proportionals, and knows to bring them into Use upon Occasion, will seldom stand in need of the particular Propositions of the Fifth Book. Only two of them, which yet are almost Axioms, may not improperly be inserted and illustrated by Examples, in way of Appendix, *because of the Frequency of their Use in all the Parts of the Mathematicks; which therefore shall be done after the Definitions.*

DEFINITIONS.

1. AN *Aliquot* Part of Magnitude, is that which being so many times more or less repeated, doth measure or is just equal to the Magnitude. An *Aliquant* Part is that which doth not measure it.

The Length of one Foot is an *Aliquot* Part of the Length of 10 Feet, because being ten times repeated it measures it. But the Length of four Feet is an *Aliquant* Part of a Line of 10 Feet, because being so many times repeated, to wit, twice, it falls short of it, but being thrice repeated it exceeds it.

2. One Magnitude is said to be a *Multiple* of another, when the lesser measures the greater, and consequently is an Aliquot Part thereof; or when the greater contains the lesser so many times precisely.

3. Proportion is the mutual Respect, as to Quantity, of two Magnitudes of the same Kind.

Therefore there are in all Proportions two Terms, of which that is called the Antecedent which is first named, or which is nam'd in the Nominative Case; the other the Consequent,

When the Antecedent and the Consequent are equal, it is called Proportion of Equality; when they are unequal, Proportion of Inequality.

4. Rational Proportion is that which is betwixt commensurable Magnitudes, and may be expressed in Numbers. Irrational Proportion, that which is betwixt Quantities incommensurable, and cannot be explicated by any Numbers.

Moreover, Commensurable Quantities are those which some common Measure measureth; Incommensurable, those which cannot be measured by any common Measure.

Fig. 1. l. 5. 5. Two Proportions (that of A to B, and that of C to F) are alike, equal or the same; when the Antecedent of one (A) doth equally or in the same Manner (that is, neither more nor less) contain its Consequent (B) as the Antecedent of the other (C) contains its Consequent (F).

Fig. 2. Or when the Antecedent of the one (A) is so often contain'd in its Consequent (B), as (C) the Antecedent of the other is in its Consequent (D).

Fig. 3. 6. Two Proportions are unlike, or one is greater than the other, when the Antecedent of one (I) doth more contain its Consequent (L), than the Antecedent of the other (O) doth contain its Consequent (Q); or when the Antecedent of one is less contain'd in its Consequent, than the Antecedent of the other in its Consequent. Fig. 4.

7. Like or *similar* Parts are those which are equally or in the same Manner contain'd in their Wholes; so that what sort of Part one is of its Whole, such a Part the other is of its Whole. Which Thing indeed is nothing else, but that the Parts bear the same Proportion to their Wholes.

Aliquot Parts are like, which do equally measure their Wholes, as if each of them be one Third or one Tenth, &c. of its Whole.

Fig. 6. 8. Magnitudes (A, B, C, D) are said to be *continually* proportional when the middle Terms (B, C) are taken twice; that is, when they are each of them a Consequent in respect of the foregoing, and an Antecedent in respect of the following.

We thus pronounce continual Proportions. A is to B, as B to C; and B is to C, as C is to D. And so on.

9. Magnitudes are *discretely* proportional when no Term is twice taken.

Fig. 1. Discrete Proportions we thus pronounce: A is to B, as C to F. When there are more than three proportional Magnitudes, if they be said to be proportional, they are always understood to be discretely so.

10. When

10. When the Magnitudes (A, B, C, D) are continually proportional, the first (A) is said to have to the third (C) a duplicate Proportion of that which it hath to the second (B): And the first (A) is said to have to the fourth (D) a triplicate Proportion of that which the same first hath to the second (B): And so forwards. Fig. 6.

[If one triplicate Proportion be equal to another duplicate Proportion, the latter simple Proportion shall be sesquiplicate, or one and a half of the former simple Proportion. Let A, B, C, D, be ∺; and a, b, c, ∺; and let A the first in the former Analogy be unto D the fourth; as (a) the first in the second Analogy is to (c) the fourth; I say that (a) is to (b) in a Proportion which is one and a half of that which A is in to B. For let F be a middle Proportional betwixt B and C: Or, which is the same thing, betwixt A and D. Because of the Equality of the Proportions of A to D, and (a) to (c), and the middle Proportionals on both Sides F and (b); It will be A : F :: a : b. But the Proportion of A to F is compounded of the entire Proportion of A to B, and of the Proportion of the same B to C halved; and consequently the Proportion of (a) to (b), which is equal to that of A to F, contains the entire Proportion of A to B, and also the same halv'd, to wit, the Proportion of B to F. But the whole Proportion, with its half, is a sesquiplicate or sesquialteral Proportion, or that which is one and a half of the other. (a) Therefore is to (b) in a Proportion sesquiplicate of that of A to B. So in Astronomy, since the Cubes of the Distances of the Planets from the Sun bear that Proportion one to another, which the Squares of their periodical Times bear; so that the triplicated Proportion of the Distances, is the same with the Duplicate one of the periodical Times; It is wont to be said, that the periodical Times are in a sesquiplicate or sesquialteral Proportion of their Distances from the Sun.]

11. Antecedent Magnitudes are said to be Homologous or like to Antecedent, and Consequent to Consequent Magnitudes. As if A is to B, as C to F; A, C, Fig. 1. and B, F, are homologous Quantities.

XII. If

XII. If a Set of Pairs of Quantities contain every one the same Proportion, that is the very Proportion also which the Sum of all the Antecedents bears to the Sum of all the Consequents.

$$\frac{20}{10}+\frac{6}{3}+\frac{8}{4}+\frac{18}{9}+\frac{14}{7}=\frac{66}{33}$$

XIX. If Parts be as Wholes, the Remainders will be also in the very same Proportion.

If 30 be to 20, as 3 to 2; 27 will be to 18 also as 30 to 20, or as 3 to 2.

The

The Elements of Euclid.

BOOK VI.

THE Doctrine of Proportions, which was generally set forth in the Fifth Book, is applied in the Sixth, to plain Figures. And those Things which are delivered in this Book are so necessary to be known, that without them no Man can penetrate into the Secrets of Geometry, and reap the sweet Fruits of the Mathematicks. Each Proposition deserves to have an Encomium annexed; so great is the Utility of all.

This Sixth Book, as has been said, begins to apply that excellent Doctrine concerning Geometrical Proportion, which was just before delivered, to divers, and those certainly, most notable Uses; and beginning with Triangles, the most simple of Figures, searches out their Sides and Areas, as they answer to one another in a certain Proportion. Then it defines proportional Lines, and the proportional Augmentations or Diminutions of Figures; and shews in what manner we may either increase or diminish them according to any Proportion given. It opens likewise the Golden Rule, or Rule of Proportion, the very chief of all Arithmetic; and demonstrates, that in a rectangle Triangle, not only the Square, but also the Pentagon, Hexagon, and in general, every regular Polygon, which is described by the Hypotenuse, is equal to the Squares, Pentagons, Hexagons, or any regular Polygons whatsoever, that are describ'd by the two Sides. It also propounds most easy and certain Principles for measuring as well Solids, as Lines and Surfaces, which are of very great Use in all Parts of the Mathematicks.

DEFI-

DEFINITIONS.

1. LIKE or similar Figures, are those which both have all the Angles equal, each to each other respectively, and the Sides which are opposed to the equal Angles, or which are betwixt them, or which are about the equal Angles, (for they all come to one) Proportional.

Fig. 7. *L.* 6. As the Triangles X, Z, will be said to be like, or similar, if the Angle A be equal to the Angle F, and the Angle B equal to the Angle I, and consequently the Angle C equal to the Angle L: And moreover, if AB be to FI, as BC to LI; and BC is to LI, as CA is to LF; and CA to LF as AB to FI; by comparing always the Sides opposite to the equal Angles. In the same manner the Likeness of all right-lin'd Figures may be explained.

Fig. 29. 2. Reciprocal Figures are when the antecedent and consequent Terms of the Proportions appear on both Sides.

As in the Parallelograms X, Z,
If AC be to CB,
As FC is to CL.

The Antecedents here are AC, and FC, of which there is one in both Figures; and the Consequents are CB, and CL; of which likewise there is one in each Figure. And therefore the Parallelograms X, Z are call'd reciprocal. Understanding the same of other Figures.

3. The Altitude of a Figure is the Perpendicular let fall from the Top to the Base. This with *Euclid* is the fourth Definition.

Fig. 1. As the Altitude of the Triangle ABC is the Perpendicular AQ which falls from the Top upon the Base BC, either within the Triangle or without, upon the Base protracted. Now the Base and Top are assumed at Pleasure.

4. Like Arches of Circles are those which have the same Proportion unto their whole Circumferences.

As if each of them be a third or fourth Part, &c. of their Circumference.

PROPO-

PROPOSITION I. Theorem.

TRIANGLES (ABC, DEF) and Parallelograms (AOPC, DQRF) which are betwixt the same Parallels, or have the same Altitude, have the same Proportion betwixt themselves as their Bases, (AC, DF.) Fig. 2.

Upon this Theorem the whole Sixth Book depends, yea, whatsoever any where is demonstrated about Figures by Proportions, whether Plain or Solid.

Let there be taken any Aliquot Part of the Base DF; *e.g.* DG one Third, and let the right Line GE be drawn: The Triangle DEG will likewise be one third Part of the Triangle DEF, as is gathered from 38. *l.* 1. Wherefore DG and the Triangle DGE are like Aliquot Parts of their Consequents *. Then let there be taken away DG from the Base AC as often as it can, as suppose six times, and let the right Lines HB, IB, KB, LB, MB, NB, be drawn. Because the Lines CH, HI, &c. are each of them equal to DG, the six Triangles CBH, HBI, &c. are each of them (*a*) equal to the Triangle DEG. Therefore as often as DG is contain'd in the Antecedent AC, so often is the Triangle DEG contain'd in the Triangle ABC. By the same Reasoning it may be shew'd, that the like Aliquot Parts whatsoever of the Consequents (the Base DF, and the Triangle DEF) are in an equal Number contain'd in the Antecedents (the Base AC, and the Triangle ABC): Therefore as the Base AC, is to the Base DF; so is the Triangle ABC, to the Triangle DEF. *Q.E.D.*

* Per defin. 7. l. 5.

(a) Per 38. l. 1.

But now because the Parallelograms AP, DR are (*b*) double to the Triangles ABC, DEF, they also will be as their Bases.

(b) Per 41. l. 1.

Corollary.

THE Triangles (ABC, FIL) and the Parallelograms which have equal or the same Bases (AC, FL), have that Proportion one to another, which their Altitudes (BO, IQ) have. Fig. 3.

For let QS, OR, be made equal to the equal Bases (FL, AC); QS, OR will then be equal. Draw SI, RB. If in the Triangles OBR, QIS, you take BO, IQ for the Bases, OR, QS, will be their Altitudes; which seeing they are equal, the Triangles OBR, QIS (a) will be betwixt themselves, as their Bases BO, IQ. But because by the Construction QR is equal to AC, and QS equal to FL, the Triangles OBR, QIS, are (b) equal to the Triangles ABC, FIL. Therefore the Triangles ABC, FIL, are also as BO is to QI.

(a) Per 1. l. 6.
(b) Per 38. l. 1.

Fig. 50. Coroll. (2) *Hence a Trapezium ABCD, whose Sides AD and BC are parallel, may be divided into any equal Parts whatsoever. For let CE be made equal to AD. Because of the Equality of the Angles vertically opposite* (c) *AFD, EFC, and of the alternate Angles* (d) *DAF, FEC, and ADF, ECF, and the Equality of the Bases AD, CE, by Construction, the Triangles ADF, FCE* (e), *are equal; and therefore the Triangle ABE is equal to the Trapezium ABCD. Therefore the Base BE being divided into any equal Parts whatsoever; as for Instance, three, BG, GR, RE, the Triangles ABG, AGR, ARE, shall each of them be one third Part of the Trapezium.* Q. E. I.

(c) Per 15. l. 1.
(d) Per 27. l. 1.
(e) Per 26. l. 1.

PROP. II. Theorem.

Fig. 4. *If to one Side of a Triangle (as BC) there be drawn (FL) a Parallel, this cuts the Sides proportionally; that is, (AF) will be to (FB) as (AL) to (LC).*

And if the right Line (FL) cuts the Sides (BA, CA) proportionally, it will be parallel to the other Side (BC).

Part 1. Let BL, CF be drawn. Because FL is supposed parallel to BC, the Triangles FBC, LCF having the same Base are (f) equal. Therefore the Triangle X, hath the same Proportion to both; now the Triangle X is to the Triangle FBL, as the same Triangle X is to that LCF. But the Triangle X is to the Triangle FBL (g), as AF is to FB; and the Triangle X is to that LCF as AL (h) is to LC. Therefore also AF is to FB, as AL to LC. *Q. E. D.*

(f) Per 37. l. 1.
(g) By the foregoing.
(h) By the same.

Part

Part 2. As AF is to FB, so is the *(a)* Triangle X to the Triangle FBL: And as AL is to LC, so is the same Triangle X to the Triangle LCF. Now AF is suppos'd to be to FB, as AL is to LC. Therefore the Triangle X is to the Triangle FBL, as the same X is to LCF. Therefore the Triangles FBL, LCF are equal. Therefore seeing they have a common Base FL, the Lines FL, BC, are *(b)* parallel. *Q. E. D.*

(a) By the foregoing.
(b) Per 39. l. 1.

Corollary.

If unto (BC) one side of a Triangle there be drawn more Parallels (IO, FL), all the Segments of the Sides will be proportional. Fig. 5.

Let FQ be drawn parallel to AC. The right Lines FS, SQ, are equal *(c)* to LO, OC. But BI is to FI, as QS to SF *(d)*. Therefore BI is also to IF, as CF to OL.

(c) Per 34. l. 1.
(d) Per 2. l. 6.

PROP. III. Theorem.

If a right Line (BF) which bisects an Angle of a Triangle, doth also cut the Base (AC), the Segments of the Base (AF, FC) will have the same Proportion betwixt themselves as the Sides (AB, BC) have. Fig. 6.

And if the Parts of the Base (AF, FC) have the same Proportion betwixt themselves, as the other Sides (AB, CB) the Line (BF) which cuts the Base, bisects the opposite Angle (ABC).

Part 1. Draw forth CB until BL be equal to BA; and join AL. Because in the Triangle Z the Sides LB, AB, are equal, the Angles also *(e)* L and O are equal. Because therefore the external Angle ABC is equal to the two internal ones *(f)* L, O, the Angle I, which by the Hypothesis is half ABC, will be equal to the Angle L. Therefore AL, FB *(g)* are parallel. Therefore in the Triangle ACL, AF is to FC *(h)* as LB (that is, AB) is to BC. *Q. E. D.*

(e) Per 5. l. 1.
(f) Per 32. l. 1.
(g) Per 29. l. 1.
(h) Per 2. l. 6.

Part

Part 2. Produce CB again until BL be equal to BA: Because AF is suppos'd to be to FC, as AB (that is, (a) Per 2. l. 6. LB) is to BC; AL, FB (*a*) are parallel. Therefore (b) Per 27. l. 1. the external Angle I is equal to the internal one L (*b*); and the alternate Q equal to the alternate O. But be- (c) Per 5. l. 1. cause LB, AB, are equal, the Angles L and O (*c*) are equal. Therefore I and Q are also equal. Therefore ABC is bisected. *Q. E. I.*

PROP. IV. Theorem.

(d) Per def. 1. l. 6. TRiangles which are equiangular to one another are like or similar, that is, have their Sides also (*d*) that are opposite to the equal Angles proportional.

Fig. 7. In the Triangles X, Z, let the Angle A be equal to the Angle F, and the Angle C to the Angle L, and the Angle B to the Angle I; I say, that AB is to FI, as AC is to FL; and AC is to FL, as CB is to LI; and CB is to LI, as BA is to FI.

Fig. 7, 8. *Demonst.* If the Angle F be laid upon its equal A, the Sides FI, FL will fall upon the Sides AB, AC. And because the external Angle AIL is by the Hypo- (e) Fig. 8. alone. thesis equal to the internal B (*e*), therefore (*f*) IL, BC, (f) Per 29. l. 6. are parallel. Therefore BI is to IA (*g*) as CL to LA: Therefore by compounding, BA is to IF, as CA to (g) Per 2. l. 6. LF. And if the Angle L be laid upon the Angle C, it will be shew'd in the same manner, that AC is to FL, as BC is to IL; and if the Angle I be laid upon the Angle B, it will be shew'd in the same manner, that BC is to IL as AB to FI. The Proposition therefore is prov'd.

Corollaries.

Fig. 8. 1. IF in a Triangle a Line LI be drawn parallel to one Side BC, the Triangle LFI will be like to the whole CBF; and consequently CF will be to LF, as BC to LI.

For since LI, BC, are parallel, the external Angles FIL, FLI will (*per* 27. *l.* 1.) be equal to the internal ones B and C: But F is common to both Triangles: There-

Therefore they are equiangular. Therefore the Sides CE, LF opposite to the equal Angles B and FIL (a) are proportional to the Sides BC, LI, which are oppos'd to the common Angle F. (a) By the foregoing.

2. If in a Triangle a right Line BF drawn from the opposite Angle B, doth cut the Parallels AC, LO, it cuts them proportionally. Fig. 9.

For by Coroll. 1. AF is to LI, as FB is to IB; and FC also is to IO, as FB is to IB. Therefore AF is to LI, as FC to IO. Therefore by changing, AF is to FC, as LI to IO.

[3. *From Coroll.* 1. *We learn to find the Height of a Tower, or any elevated Point, by only the Shadow of a Staff. Fix the Staff FL perpendicularly upon the Ground in that Place where the Ray of the Sun KBA, that terminates the Shadow of the Tower BZ, may also pass thro' L. There will be in the Triangle AZB, the Line FL parallel to the Height of the Tower ZB. Whence as AF, the Distance of the Staff from the Point of the Shadow, is to FL, the Length of the Staff; so is AZ, the Distance of the Tower from the Point of the Shadow, to ZB the Height of the Tower. And because the three first Terms are easily had by measuring, the fourth,* i. e. *the Height of the Tower is had also.* Q. E. I. Fig. 51.

4. *From this also incomparably useful Proposition, we may deduce that Famous Theorem of* Ptolomy; *to wit, That in every Quadrilateral inscrib'd in a Circle, the Rectangle of the Diagonals* $AC \times BD$ *is equal to the two Rectangles of the opposite Sides,* $AB \times CD$ *and* $AD \times BC$. *For let the Angle BAE be made equal to the Angle CAD. Because the Angles BAE, CAD, are equal by Construction, the Angles ABE, ACD, standing upon the same Arch AD, are * equal; therefore the Triangles BAE, CAD, are alike. And* $AC : CD :: AB : BE$; *and consequently † the Rectangle of the Extremes* $AC \times BE$ *is equal to the Rectangle of the Means* $CD \times AB$. *In like manner, because the Angle EAD is equal to BAC by Construction, and the Angles ADE, ACB, as standing upon the same Arch AB, are equal: The Triangles ADE, ACB, will be like; and* $AD : DE :: AC : CB$. *And therefore the Rectangle of the Extremes* $AD \times CB$, *is equal to the Rectangle of the Means* $DE \times AC$. *But the Rectangles* Fig. 52. * Per 21. l. 3. † Per 16. l. 6.

AC × BE, and AC × DF, are equal to the Rectangle AC × BD. Therefore the Rectangles AB × DC, and AD × BC, which are made by the opposite Sides, are equal to the Rectangle AC × BD, which is made by the Diagonals. Q. E. D.]

PROP. V. Theorem.

Fig. 10. IF *two Triangles have all their Sides mutually proportional, they shall also be mutually equiangular.*

That is, If AB be to RF, as AC to RQ; and as AC is to RQ, so is CB to QF; and as CB is to QF, so is AB to RF; I say, that the Angles opposite to the Antecedents, are equal to the Angles opposite to the Consequents; to wit, C to I, and B to F, and A to O.

Ang.	Antec.	Conseq.	Ang.
C	AB	RF	I
B	AC	RQ	F
A	CB	QF	O.

Make X and Z equal to A and C; and let the Sides meet in N. The Angles B and N will (*a*) be also equal. Because therefore the Triangles P, T, are equiangular, AB (by the foregoing) will be to RN, as AC to RQ. But by the Hypothesis, AB is to RF, as AC to RQ. Therefore AB is to RF, as the same AB is to RN. Therefore RN, RF are equal. In the like manner I might shew that QN and QF are equal. Therefore the Triangles T, S, are equilateral to each other. Therefore the Angles I, F, O, are equal (*per* 8. *l.* 1.) to the Angles Z, N, X, that is, by the Construction to the Angles C, B, A. *Q. E. D.*

(a) *Per Corol.* 9. p. 32. l. 1.

PROP. VI. Theorem.

Fig. 10. IF *two Triangles (P, S) have one Angle (A) equal to one Angle (O); and the Sides (AB, AC, RF, RQ) which contain the equal Angles proportional; the Triangles will be similar.*

Let

Let X and Z be made equal to the Angles A, C, and the Sides meet together in N. Therefore the Angles B and N will (†) be alſo equal. Then it may be ſhew'd, as in the foregoing, that R F, R N, are equal. But R Q is common to both Triangles S, T. The Angles alſo O and X are equal, becauſe they are both equal to the ſame A; the one X by the Conſtruction, and O by the Hypotheſis. Therefore (*a*) I and F are likewiſe equal to Z and N. Therefore the Triangle S is equiangular to the Triangle T; that is, by the Conſtruction, to the Triangle P. Therefore S, P, are ſimilar (*per* 4. *l*. 6.) *Q. E. D.*

† *Per Corol.* 9. *p.* 32. *l.* 1.

(a) *Per* 4. *l.* 1.

PROP. VII.

I*S ſcarce of any Uſe.*

PROP. VIII. Theorem.

IN a Rectangle Triangle, the Perpendicular (B C) *let down from the right Angle to the Baſe, cuts the Triangle into Parts ſimilar to the whole, and betwixt themſelves.* *Fig.* 11.

In the Triangles A B F and L, the Angle F is common, but the Angles A B F and X are by the Hypotheſis right ones, and conſequently equal. Therefore the other Angles A and O are (*b*) alſo equal. Therefore (*c*) the Triangles A B F and L are like. In the ſame manner the Triangles A B F and R may be ſhew'd to be equal, and the Angle I equal to the Angle F. From which it is now manifeſt, that R and L alſo are like, ſeeing the Angles I and F; O and A; U and X are equal. *Q. E. D.*

(b) *Per Corol.* 9. *p.* 42. *l.* 1.
(c) *Per* 4. *l.* 6.

Corollaries.

FIrſt, B C is a mean Proportional betwixt A C, and C F.

For seeing there be in the Triangles R and L,

equal Ang. I. F	equal Ang. A. O
Sides oppos. A C. C B.	Sid. oppos. C B. C F.

(a) Per 4. l. 6. It is manifest (*a*) that A C : C B :: C B : C F.

2. B F is a mean Proportional betwixt A F, and C F. Likewise A B a mean betwixt F A and C A.

For in the Triangles A B F and L,

equal Ang. A B F. X.	equ. Ang. A. O
Sides oppos. A F. B F	Sid. oppos. B F. C F

(b) By the same. Therefore A F (*b*) : B F :: B F : C F. Likewise because in the Triangles A B F and R there be

equal Ang. A B F. V.	equ. Ang. F. I
Sides oppos. A F. A B	Sid. oppos. A B. A C

It will be again A F : A B :: A B : A C.

Fig. 11. 3. *Hence we learn to measure an inaccessible Line, one Term whereof is accessible. Let the inaccessible Line be C F. Let there be rais'd from the Point C the Perpendicular C B: And to any Point of this Perpendicular as B, let there be applied a Square or any right Angle A B F; so that in looking along the Line B F the Point F, and along the Side B A the Point A may be observed. Let the accessible Line A C be measured, and from the following Analogy the inaccessible C F will be made known. A C : C B :: C B : C F. Let the Square then of the Line C B be divided by the Line A C, and the Quotient* (c) *will give the sought Line C F.* Q. E. I.

(c) Per Corol. 3. p. 17. l. 6.

PROP. IX. Problem.

Fig. 12. TO *divide a given Line (A B) according to a given Proportion (F I to I L).*

Let the infinite Line A Z be drawn. From which take A Q, Q R, equal to F I, I L. From R draw R B. Parallel to this draw Q C from Q. I say the thing is done.

It is manifest from *prop.* 2. *l.* 6.

PROP.

PROP. X. Problem.

To divide a given Line as (A B) in like manner as another given one (A I) hath been divided (in F, C). Fig. 13.

Let the right Line I B join the Extremities of the two Lines. Draw Parallels to this from the Points F, C, which may meet the right Line, that is to be cut, A B in L and Q. I say the Thing is done.

This is manifest from the Corollary of *Prop. 2. l. 6.*

[*Or thus, if the cut Line I A be greater than that which is to be cut B Q, let three Circles touching one another be describ'd with the Diameters IF, IC, IA; and let the Subtense B Q be fitted from the Point I to the Circumference of the greatest Circle: The two lesser Circles will cut the Line B Q in the Points L, P, in the Proportion * of the Sections of the Diameter L A. If the Line I A be cut into four Parts, four Circles are to be drawn; if into five, then five Circles; and so infinitely.*] Fig. 53. * Per Corol. 4. p. 31. l. 3.

Scholium.

FROM this Proposition we learn to cut a right Line given into any equal Parts whatsoever. Let an infinite right Line make any Angle with the right Line to be cut A B; from which take with a Pair of Compasses so many equal Parts A C, C F, F I, as you would divide A B into. Draw the right Line I B, and the Parallels to it F L, C Q. I say the Thing is done. Fig. 13.

We may do the same Thing otherwise, and more easily after *Maurolycus*, in the manner following. Let A B be to be trisected or divided into three equal Parts. Draw the infinite Line I X parallel to A B, above or below it. From I X, if it be below A B, take with a Pair of Compasses three equal Parts I Q, Q R, R S, which together may be greater than A B; but lesser if I X is above. Thro' I and A, as likewise thro' S and B draw right Lines which may meet together in C. From C to Q and R draw right Lines: These will trisect the given Fig. 14.

Line A B. The Demonstration appears from *Coroll.* 2. *Prop.* 4.

Fig. 15. Again with *Maurolycus*, we may otherwise obtain the same thing, to wit, thus: Let A B be to be quadrisected. Draw the infinite Line A X and B Z also an infinite Line parallel to it. From these take with the Compasses equal Parts A L, L O, O Q, and B V, V S, S R, in each fewer Parts by one than are required in A B; then let there be drawn the right Lines, L R, O S, Q V. These will quadrisect the given A B.

For because by Construction, the Lines L O, R S, parallel and equal, are join'd by L R and O S, these also (*a*) will be parallel. In the like manner O S and Q V are parallel. Therefore seeing A Q is cut into three equal Parts, A I will also (*b*) be cut into so many equal Parts. Likewise B C will be cut into three equal Parts. Therefore the whole A B will be cut into four equal Parts.

(a) Per 33. l. 1.
(b) Per Corol. prop. 2. l. 6.

These two Ways of Practice are easier than *Euclid*'s, because fewer Parallels are to be drawn.

PROP. XI. Problem.

Fig. 16. *To find a third Proportional to two right Lines given (A B, B C).*

Draw the right Line A C. From B A produc'd take A F equal to B C. Thro' F draw the infinite Line F X parallel to A C, which infinite Line let B C produc'd meet in L. I say that A B is to B C, as B C to C L.

For A B : A F (*c*) :: B C : C L. But A F (*d*) is equal to B C. Therefore A B : B C :: B C : C L; and so C L is the third Proportional sought.

(c) Per 2. l. 6.
(d) By the Construction

Otherwise,

Fig. 17. LET A B and B C be set at a right Angle. Join A C. From C draw C X perpendicular to A C infinite; which C X let A B produc'd meet in L. I say A B : B C :: B C : B L. It is manifest from *Coroll.* 1. *pr.* 8.

Scho-

Scholium.

A Given Proportion may not only be continu'd in three, but alſo in infinite Terms, and the whole Sum of the infinite proportional Terms be exhibited. *Gregory* of *St. Vincent* hath very handſomely proſecuted this Matter, and the whole Buſineſs of Geometrical Progreſſion in the whole ſecond Book of his Work. We for the ſake of the Studious, will here preſent ſuccinctly the Conſtruction and Demonſtrations of the Thing propoſed.

Problem.

LET a Proportion of the greater Inequality be given, as A B to B C. It is requir'd to continue this thro' infinite Terms, and to preſent the Sum of them all. *Fig.* 19.

Let the Perpendiculars A L, B O, be erected, and taken equal to the given Lines A B, B C, and thro' L, O, let a right Line be drawn, meeting with A B C produc'd in Z. I ſay, 1. If from C you erect the Perpendicular C Q; C Q ſhall be a third Proportional. Transfer Q C into C E, and from E erect E R; this ſhall be a fourth Proportional. Transfer E R into E F, and erect F S; this ſhall be a fifth Proportional; and ſo the Proportion of A B, to B C, that is, of A L to B O, will be continu'd thro' the Terms, A L, B O, C Q, E R, F S, &c. or A B, B C, C E, E F, &c. infinitely, becauſe every Term (as F S) may be taken away from the remaining one F Z; for ſeeing L A (that is, A B) is leſs than A Z; F S alſo (a) muſt ever be leſs than F Z.

(a) *Per Corol.* 1. *prop.* 4. *l.* 6.

I ſay (2.) A Z is equal to the whole Sum of the infinite Proportionals.

Part I. [*It being ſuppos'd as before,* $AZ : BZ :: AB : BC$; *it will be by alternating* $AZ : AB :: BZ : BC$. *And by dividing,* $AZ - AB : AB :: BZ - BC : BC$; *that is,* $BZ : AB :: CZ : BC$. *Therefore by inverting* $AB : BZ :: BC : CZ$. *And by compounding* $AB + BZ :$

I 4

$BZ::BC+CZ:CZ$; *that is*, $AZ:BZ::BZ:CZ$] But as A Z is to B Z, so is L A to O B; and as B Z is to C Z, so is O B to Q C. Therefore also L A is to O B, as O B is to Q C. In the same manner I might shew that O B is to Q C, as Q C to R E; and so forwards infinitely.

Part 2. The whole Sum of the infinite Terms is neither less than A Z, nor greater; therefore it is equal. It is not greater, because seeing we have shew'd above, that Q C is lesser than C Z, and R E than E Z, and S F than F Z, and so on infinitely, all the Terms Q C, R E, S F, &c. may be infinitely set one by another in the right Line A Z; so that the Point Z shall never be reach'd. Again, the said Sum will not be less, because I have above shew'd, A Z, B Z, C Z, to be continually proportional; and in the same manner the same Thing is shew'd of the rest E Z, F Z, &c. Seeing therefore by transferring the Proportionals Q C, E R, F S, &c. into C E, E F, F I, the Remainders E Z, F Z, I Z, &c. are always continually proportional, as we have already shewed; we shall at the last come unto a Remainder less than any given one; and therefore the Sum of the Proportionals shall exceed every Quantity that is less than A Z; from whence it self cannot be less than A Z.

Seeing therefore it is neither greater nor less than A Z, it shall be equal to it. *Q. E. D.*

Theorem.

THE Difference of the first Terms, the first Term, and the whole Sum of the infinite Proportionals, are continually proportional.

Fig. 19. In the upper Figure let O X be drawn parallel to A Z. Therefore L X shall be the Difference of the first Term A L or A B, and of the second B O, or B C. Because X O is parallel to A Z; L X shall be to X O, as (*a*) L A is to A Z. But X O is A B, and X A likewise is A B. Therefore the Difference L X is to the first Term A B, as A B the first Term is to A Z the whole Sum. *Q. E. D.*

(a) *Per Coroll.* 1. *prop.* 4. *l.* 6.

Fig. 20. The same Thing may be demonstrated universally and very briefly in every kind of Quantity, thus: Let there be any continual Proportionals whatsoever (as well Numbers, as other Quantities) A Z, B Z, C Z, &c. and let

let them all be transfert'd upon the first AZ. Therefore AB, BC, CE, EF, &c. will be the Differences of the Proportionals; which, together with the last Quantity FZ are equal to the first AZ. Now because if Proportionals be continued infinitely, the last Quantity vanisheth away, it is manifest that the Differences of the infinite Proportionals are equal to the first AZ. Then because AZ is to BZ, as BZ is to CZ, and so on: By dividing, AB will be to BZ, as BC to CZ; and by converting, as AB, the first Difference, is to AZ, the first Quantity; so BC, the second Difference, is to BZ, the second Quantity, and so forwards. Therefore as AB, the first Difference, is to AZ the first Quantity; So all the Differences, (that is, as I have already shew'd, the first Quantity AZ) are to all the Quantities, that is, to the whole Sum of the infinite Quantities. *Q. E. D.*

PROP. XII. Problem.

Three right Lines being given (AB, BC, AF) to find a fourth Proportional. Fig. 21.

Let the two right Lines be dispos'd, as the Figure shews, and draw the right Line BF, to which let the infinite Line CZ be made parallel. Let AF produc'd to L meet CZ.

I say, AB is to BC, as AF to FL, as is manifest from *Prop.* 2. of this Book. Therefore FL is the fourth Proportional sought.

Scholium.

OUR Countryman *Bettin* in his Treasury of Mathematical Philosophy, doth handsomely from 35. l. 3. and 14 of this, which depends not upon the present Proposition, find out a fourth Proportional, three being given, and a third two being given, after this manner.

If three right Lines be given, let the second CB, and the third BD be join'd right to one another, so as to make one right Line, and let the first BA touch them Fig. 22.

in

in the Point B, in what Angle you will. Thro' the (a) Per 5. l. 4. Points C, A, D, describe a Circle (*a*), which let AB the first Line meet in the Point Z. BZ is a fourth Proportional.

(b) Per 35. l. 3. For seeing the Rectangles ABZ, CBD, are (*b*) equal, AB will be to BC, as BD to BZ, by the 14th of this Book, which, as was said, depends not upon this.

Fig. 23. If there be given two right Lines AB, BC; let BD equal to BC be join'd to BC, so as to make one strait Line. Then let the first AB touch BC in B in any Angle. Then the rest is as before, and BZ will be the third Proportional sought.

The Demonstration is the same; for seeing the Rectangles ABZ, CBD, are equal, AB will be to BC, as BD (that is, BC) is to BZ.

PROP. XIII. Problem.

Fig. 24. *TWO right Lines given (AC, CB) to find a mean Proportional.*

Let the whole compound Line AB be bisected in O, and from the Centre O a Circle be described thro' A and B; from C erect a Perpendicular CF, meeting the Circumference in F.

I say, AC is to CF, as CF is to CB.

(c) Per 31. l. 3. For let AF, BF be drawn; the Triangle (*c*) AFB is right-angled, and from the right Angle there is drawn the Perpendicular FC to the Base. Therefore AC is (d) Per Coroll. 1. p. 8. l. 6. to CF as (*d*) CF is to CB.

Corollary.

HEnce it is manifest, that if from any Point of the Circumference (as F) there be drawn a Perpendicular (FC) to the Diameter, this Perpendicular is a mean proportional betwixt the Segments of the Diameter (AC, CB).

Scholium.

Scholium.

THIS Place requires, that we ſhould ſay ſomething briefly concerning the finding out of two mean Proportionals betwixt two given Lines. All the Geometricians of *Greece*, at *Plato*'s Suggeſtion, ſet themſelves with all their Might to the Solution of this Problem. Divers moſt ſubtle Ways of Practice are recited by *Eutocius* in his Commentary on *Archimedes*; as thoſe of *Plato*, *Archytas* the *Tarentine*, *Menæchmus*, *Eratoſthenes*, *Philo Byzantius*, *Hero*, *Apollonius* of *Perga*, *Nicomedes*, *Diocles*, *Sporus*, *Pappus*; to whom the later Times have added *Verner*, *Gregory* of *St. Vincent*, *Renatus Carteſius*. Out of all theſe we ſhall ſelect Three more eaſy than the reſt.

Plato's *Method.*

IT is requir'd to find out two Means betwixt the given Lines A B, B C. *Fig. 25.*

Let A B, B C be ſet in a right Angle, and be produc'd infinitely towards X and Z. Then let two Squares (ſo our *Claudius Richards* hath it; for *Plato* himſelf made uſe of one Square only, but which had inſerted into its Side *D E a Rule movable along D E,) let two Squares, I ſay be taken, and the Angle D of one Square be applied to the right Line B X, in ſuch certain wiſe, that one Side may alſo paſs thro' A; and to the Point E in which the other Side cuts the right Line B Z, let a ſecond Square be applied, which will paſs thro' C. I ſay, that B D, B E, are two Means betwixt the given Lines A B, B C; that is, as A B is to B D, ſo is B D to B E, and B E to B C. * *See Fig. 26.*

The Demonſtration is manifeſt from *Coroll.* 1. *Prop.* 8. *l.* 6. for A D E is a right-angled Triangle, and from the right Angle to the Baſe there falls the Perpendicular D B. Therefore by the ſaid Corollary, as A B is to B D, ſo is B D to B E; and for the ſame Cauſe, as B D to B E, ſo is B E to B C. Therefore betwixt the given right Lines A B, B C, there are found two mean Proportionals B D, B E. Which was the Thing to be done. This manner of ſolving the Problem is the eaſieſt of all to be underſtood.

The

The Method of Philo *the* Byzantine.

Fig. 27. LET the two given right Lines A B, B C, be set together at a right Angle; then let the Rectangle A B C D be perfected, and let D A, D C be produc'd infinitely, and let the Diameters B D, A C be drawn, cutting each other in E. From the Centre E thro' B let a Circle be drawn, which, because A B C is a right Angle (*a*) will pass thro' A and C. Then let a Rule be applied to the Point B, so that the intercepted right Lines B G, O F, may be equal. I say, that A F, G C, are two mean Proportionals betwixt the given A B, B C; that is, as A B is to A F, so is A F to G C, and G C to C B.

(a) *Per* 31. l. 3.

Demonst. Because G B, O F (*b*) are equal, O G, B F, will be also equal. Therefore the Rectangles O G B, B F O, that is (*c*) the Rectangles D G C, D F A, are equal. Therefore as G D is to D F, so (*d*) reciprocally A F is to G C, but G D is to D F (*e*) as B A to A F. Therefore as B A is to A F, so A F is to G C. Again, I have already shew'd that A F is to G C, as B A is to A F; but B A is to A F, as G D is to D F; that is, as G C is to C B; therefore A F will also be to G C as G C is to C B. Therefore all four, B A, A F, G C, C B, are continually proportional; and therefore betwixt the given Lines A B, B C, two Means have been found. *Q. E. I.*

(b) By the Construction.
(c) *Per Corol.* 1. *p.* 36. *l.* 3.
(d) *Per* 14. *l.* 6.
(e) *Per Corol.* 1. *p.* 4. *l.* 6.

These two Methods of Solution, altho they be ingenious and easy enough; yet because a due Application of a Square and Rule is not made but by trying, they are not Geometrical.

The Method of Cartes.

Fig. 28. LET an Instrument of such sort be provided; that two Rules may be open'd and shut about Y. Let there be inserted into these divers Squares connected together betwixt themselves in the Points B, C, D, E, F, G, in such sort that in the mean while that the Rules Y X and Y Z are open'd, the Square B C may impel the Square C D in the Rule Y Z, and the Square C D may impel

impel the Square D E in the Rule Y X, and the Square D E may impel F E, and E F impel or force forward F G and so on: But so that while the Rules X Y and Y Z are shut, all the Points B, C, D, E, F, G, tend to fall upon one and the same Point A. By this Instrument not only two, but also four and six, yea, as many Means as you will, betwixt two given right Lines may be found. Which thing can be obtain'd neither by the Sections of a Cone, nor by any Methods found out by the abovesaid Authors.

For two Means three Squares are requir'd; for four Means five Squares, and so on.

Let the lesser of the given right Lines be transferr'd upon the Rule Y X, and let it be Y B; the greater upon the Rule Y Z, and let it be Y E. Let the first Square be applied to the Point B, and be fixed there, and let the Rules be open'd, until the Side of the third Square passeth thro' E. I say, that Y C, Y D, are two Means betwixt the given Y B, Y E; that is, that Y B is to Y C, as Y C is to Y D, and Y D to Y E.

The Demonstration appears out of *Caroll.* 2. *pr.* 8. *l.* 6. For from the Nature of the Instrument, in the Triangle Y C D, the Angle at C is a right one, and from it C B falls perpendicular upon the Base Y D. Therefore by the said Corollary, as Y B is to Y C, so is Y C to Y D. Again, because in the Triangle Y D E, the Angle at D is a right one, and from it there falls the Perpendicular D C upon the Base Y E, as Y C is to Y D, so is Y D to Y E. Therefore Y B, Y C, Y D, Y E are four continual Proportionals. Betwixt the given Lines therefore Y B, Y E, there have been found two mean Proportionals Y C, Y D. *Q. E. I.*

If betwixt the given ones Y B, Y G, there be required four Means, open the Rules, until the Side of the fifth Rule F G passeth thro' G. There will be Y C, Y D, Y E, Y F, four Means betwixt Y B, Y G. The Demonstration is manifest from the said Corollary.

This way, altho' the Instrument is more operose than *Plato's*, is truly an excellent one; both because it doth nothing by bare Trial, and because it extends it self unto four and six, and as many Means as you will.

The Deliacal Problem, to wit, the Duplication of the Cube, is performed by two Means, and all Bodies whatsoever

soever are increas'd or diminish'd in a given Proportion (a) by the same Method; like as the same Thing is perform'd in plain Figures (b) by one Mean. *Hippocrates* first open'd this way, which as the singular and only one, all Geometricians that have follow'd him have embrac'd.

(a) See schol. pr. 18. l. 12.
(b) Corol. 3. pr. 20. l. 6.

PROP. XIV. Theorem.

Fig. 29, 30.

Equal Parallelograms (X, Z) which have one Angle (C) equal to one (O); have their Sides also, which are about the equal Angles, reciprocal; (that is, AC is to CB, as FO is to OL).

And if they have the Sides thus reciprocal, the Parallelograms are equal.

Part I. Let IL and SB being produc'd meet together in Q. The Parallelogram X is to the Parallelogram R, as AC is to CB (c); and Z is to R (d), as FO to OL. But because by the Hypothesis X and Z are equal, X is to R as Z is to R. Therefore also AC is to CB as FO is to OL. *Q. E. D.*

(c) Per 1. l. 6.
(d) By the same.

Part II. As AC is to CB, so X is to R (e): And as FO is to OL, so is Z to R. But already by the Hypothesis AC is to CB, as FO to OL. Therefore X is to R, as Z is to R. Therefore X and R are equal. *Q. E. D.*

(e) By the same.

[Coroll. *On this depends the Demonstration of the inverse Rule of Proportion. For in it there is always some Rectangle given as X; and one Side of another equal Rectangle, as CB; and the other Side is sought. As therefore AC the first Side of the given Rectangle is to CB, the given Side of the other Rectangle; so reciprocally FC the sought Side is to CL the second Side of the given Rectangle. The Rectangle therefore CB×FC is equal to the Rectangle AC×CL: And the latter Rectangle given being divided by the given Side of the former CB, the Quotient will give the sought Side FC.* Q. E. I.]

PROP.

PROP. XV. Theorem.

EQual Triangles (*ACL*, *FCB*) *which have one Angle (C) equal to one (O) have also their Sides about the equal Angles reciprocal (that is, AC is to CB, as FO to OL).* Fig. 31, 32.

And if they have their Sides thus reciprocal, the Triangles are equal.

Let the right Line LB be drawn; the rest of the Demonstration is the same as that of the foregoing.

Corollary.

AS well Parallelograms as Triangles, which have their Bases and Altitudes reciprocal, are equal: And so conversly.

It is manifest from the two foregoing Propositions.

PROP. XVI. Theorem.

IF *four right Lines (AB, FI; IL, BC) be proportional, (that is, if AB be to FI, as IL is to BC) the Rectangle (X) under the Extremes (AB, BC) is equal to the Rectangle (Z) under the Means (FI, IL).* Fig. 33.

And if the Rectangle under the Extremes be equal to the Rectangle under the Means, those four right Lines will be proportional.

Part I. In the Rectangles X and Z, about the right and therefore equal Angles BI, by the Hypothesis AB is to FI, as reciprocally IL to CB. Therefore X and Z (*a*) are equal. *Q. E. D.* (a) Per 14. l. 6.

Part II. Because X and Z are now suppos'd equal; therefore, (*b*) about the equal Angles B and I, AB is to FI as reciprocally IL to BC. *Q. E. D.* (b) By the same.

[Coroll. (1.) *Hence it is easy to apply the given Rectangle Z (c) to the given right Line AB; to wit, by making AB : FI :: IL : BC. For BC is the Rectangle Z applied to the given right Line AB.*] (c) Per 12. l. 6.

[Coroll

[Coroll. (2.) *Upon this Proposition depends the Demonstration of the direct Rule of Proportion. For in it there is always given some Rectangle, as CL: And another like Rectangle is sought, one Side whereof is also given. It will therefore be, as BC the first Side of the Rectangle given, is to EO the Side of the Rectangle sought; so directly CE, the second Side of the Rectangle given, is to OA the other sought Side. Therefore the Rectangle CE×EO is equal to the Rectangle BC×OA. And the Rectangle CE×EO being divided by BC, the Quotient, will give OA the other Side which was sought.* Q. E. I.]

PROP. XVII. Theorem.

Fig. 34. *If the right Lines (AB, FL, BC) be proportional, the Rectangle under the Extremes (AB, BC) shall be equal to the Square of the Mean (FL).*

And if the Rectangle under the Extremes be equal to the Square of the Mean, those three right Lines are proportional.

Part I. Let O be taken equal to the Mean FL. Because therefore by the Hypothesis AB is to FL, as FL to BC, and O is equal to FL; AB will also be to FL, as O is to BC. Therefore (a) the Rectangle under the Extremes AB, BC, is equal to the Rectangle under the Means FL and O, that is, is equal to the Square of FL.

(a) By the foregoing.

Part II. This is demonstrated in like manner from the second Part of the foregoing.

Corollary.

Fig. 24. From this, taken together with the 13th, it is manifest, that if in a Circle FC be perpendicular to the Diameter, the Rectangle ACB is equal to the Square of FC.

[(2.) *If A×B be equal to the Square of C; then A:C::C:B.*

(3.) *If A:C::C:B; and Cq be divided by A, the Quotient (b) will be B.*]

(b) Per Corol. 2. p. 16. l. 6.

I PROP.

PROP. XVIII. Problem.

Upon a given right Line (RS) to describe a Polygon like, and in like manner posited to a given one (BQ). Fig. 35.

Resolve the given Polygon BQ into Triangles. Upon the given right Line RS make the Angles (a) R, O, equal to the Angles B, A. The Sides then will meet together in X. Upon XS make the Angles V, I, equal to the Angles T, C. The Sides will then meet together in Z. I say the Thing is done. (a) *Per* 23. *l.* 1.

For because the Angles R, O, are equal to the Angles B, A, the Angles E, K, must also be equal *(per Coroll. 9. pr. 32. l. 1.)*; and because also by the Construction, V is equal to T, the whole EV must be equal to the whole KT. In like manner, because O, I, are equal to A, C, respectively, the whole Angles OI, AC, must be equal. And because V and I also are equal to T and C by the Construction, Z and Q likewise must be equal *(per Coroll. 9. pr. 32. l. 1.)* to T and C. Therefore the Polygons RZ, BQ, are mutually equiangular. It remains, that we shew that their Sides also are proportional. RS is to BF * as SX to FL; and again, SX is to FL (b), as SZ to FQ. Therefore *ex æquo* RS is to SZ, as BF to FQ, &c. * *Per* 4. *l.* 6. (b) By the same.

Coroll. *Hence is derived the Method of making Maps or Charts, whether Geographical, or Chorographical, or those which Surveyors of Land make; and of framing Ichnographical Delineations of Fields, Buildings, Countries: For they are nothing else but the Reduction of great Figures unto like Figures which are of a small Compass, which is performed by the means of this Proposition.*

PROP. XIX. Theorem.

The Proportion of like Triangles (X, Z) is duplicate of the Proportion of their Sides (AC, FI) which are subtended to the equal Angles. Fig. 36, 37.

* Per 11. l. 6. That is, if it be made * as AC is to FI, so is FI to a third AQ; the Triangle X is to Triangle Z, as AC the first to the third Proportional AQ. See *Defin.* 10. 5.

Because the Triangles X, Z are like, BA will be to (a) Per 4. l. 6. LI (*a*) as AC is to IF. But by the Construction, as AC is to IF, so is IF to AQ. Therefore also BA is to (b) Per 15. l. 6. LI, (*b*) as IF to AQ. Therefore in the Triangles QBA and Z, the Sides about the Angles A, I, (which by the Definition of like Triangles are equal) are reci- (c) Per 1. l. 6. procal. Therefore QBA and Z are equal (*c*). But the Triangle X is to QBA, as the Base AC to the Base (d) Per 1. l. 6. AQ (*d*). Therefore X is to Z, as AC to AQ. *Q.E.D.*

Coroll. *Hence is their Error to be corrected, who think that like Figures are in the same Proportion to one another, that their Sides are. For if of two, not only like Triangles, but also Squares, Pentagons, Hexagons,* &c. *yea, and Circles also, the Sides or Diameters be betwixt themselves as 2 to 1, the Figures or Areas themselves are as 4 to 1: If the Sides be betwixt themselves as 3 to 1, the Figures themselves or Areas are as 9 to 1; to wit, in a duplicate Proportion of those Sides.*

PROP. XX. Theorem.

Fig. 38. *LIKE Polygons (ABCDE, FGHIK) are divided, (1.) into like Triangles (P, S, and Q, T, and R, V) in Number equal: (2.) And proportional to the Wholes: And (3.) the Proportion of the Polygons is duplicate to that of the Sides, (AB, FG) which are betwixt the equal Angles (B, G, and BAE, GFK).*

Part I. Because the Polygons are alike, they are mutually (*per Defin.* 1. *l.* 6.) equiangular, and their Angles equal, BAE to GFK, and B to G, and BCD to GHI, and CDE to HIK, and E to K. Because (e) By the same. therefore AB is to BC (*e*) as FG to GH, and the An- (f) Per 6. l. 6. gles B and G are equal, the Triangles P, S, (*f*) are like. In like manner it will be demonstrated that R and V are like. Then because the Wholes BCD, GHI, and the subducted ones BCA, GHF, are equal, the remaining ones also, ACD, FHI, are equal. In the same manner

I might shew that A D C, F I H, are equal. Therefore (*per Corol.* 9. *pr.* 32. *l.* 1.) the third C A D is equal to the third H F I. Where also (*a*) the Triangles Q and T are alike. The first Part therefore is manifest. (a) Per 4. l. 6.

Part II. Because P and S are alike, the Proportion of P to S is duplicate to that of (*b*) C A to H F. But for the same Cause also the Proportion of Q to T is duplicate to the Proportion of C A to H F. Therefore P is to S as Q to T. In the same manner I will shew that as Q is to T, so R is to V. Therefore as one Antecedent P is to one Consequent S, so all the Antecedents P, Q, R, taken together, are to all the Consequents S, T, V, taken together, that is, so is Polygon to Polygon. Which was the other Part. (b) By the foregoing.

Part III. The Proportion of P to S is duplicate (*c*) to that of A B to F G. But the Proportion of Polygon to Polygon is the same with the Proportion of P to S, as I have already shew'd. Therefore also the Proportion of Polygon to Polygon is duplicate to the Proportion of A B to G F. Which was the third Part. (c) By the foregoing.

Corollaries.

1. ALL ordinate or regular Figures, as Squares, equilateral Triangles, Pentagons, &c. are betwixt themselves in the duplicate Proportion of the Sides. For all regular Figures are like, as is manifest from *Defin.* 1. 6.

2. If in any like Figures whatsoever, the Sides A B, F G, which are placed betwixt equal Angles, be known, the Proportion of the Figures is also known. As for example, Let A B be of two Feet, and F G of six Feet; and as 2 is to 6, so let 6 be to some other Number; to wit, 18. The lesser Figure is to the greater, as 2 is to 18, or as 1 is to 9. Now a third proportional Number is found, if (*per Corol.* 3. *pr.* 17. *l.* 6.) the second of the given ones be multiplied by it self, and the Product divided by the first. Fig. 38.

3. From the same Proposition is drawn the excellent Method of increasing or diminishing any rectilinear Figure in a given Proportion. As if I would make a Pentagon, whose Side is A B fivefold of another. Find a Mean proportional BX (*d*) betwixt the Terms of the Fig. 39. (d) Per 13. l. 6.

(a) Per 18. l. 6. Proportion given, AB, BC; upon this frame (*a*) a Pentagon like to the given one. This shall be quintuple of the given one.

For by the 20. the Pentagon AB is to BX, which is like to it, as AB the first is to BC the third Proportional.

Moreover, seeing the Proportion of Circles also is duplicate to the Proportion of their Diameter, as will be shew'd, *p*. 2. *l*. 12. This Practice belongs likewise to Circles.

Fig. 41. [Schol. *Seeing the Proportion of the Squares E, K, is duplicate of the Proportion of their Sides OR, SV; from thence the duplicate Proportion of the Sides OR, SV is wont commonly to be express'd by the Proportion of ORq to SVq.*]

PROP. XXI. Theorem.

Fig. 40. *Figures (A, B) which are like to the same (C) are also like betwixt themselves.*

This is manifest from *Defin*. 1. *l*. 6. and from *Axiom* 1. *l*. 1.

PROP. XXII. Theorem.

Fig. 40, 41. *If four or more right Lines (FI, LQ, and OR, SV) be proportional; like Figures, and in like Sort described by them (A, B, and E, K) must also be proportional.*

And conversely.

The Demonstration of the first Part is manifest. For because the Proportions of A to B and E to K are duplicate to the Proportions of FI to LQ, and OR to SV, which are by Hypothesis equal; themselves also must be equal.

The second Part is manifest also.

Fig. 24. [Coroll. *If the right Line AB be cut in any manner in C; the Rectangle contain'd under the Parts AC, CB, is a Mean proportional betwixt their Squares. Likewise the Rectangle contain'd under the Whole AB,*

I and

and one Part AC or CB is a Mean proportional betwixt the Square of the whole AB, and the Square of the said Part AC or CB. For (per Coroll. 1. p. 8. l. 6.) *it is manifest that AC:CF::CF:CB. Therefore AC Square : CF Square :: CF Square : CB Square. That is,* * *AC Square : Rectangle ACB :: Rectangle ACB : CB Square.* Q. E. D. * Per 17. l. 6.

Moreover, (per Coroll. 2. p. 8. l. 6.) *BA : AF :: AF : AC. Therefore BAq : AFq :: AFq : ACq. That is,* † *BAq : BAC Rectangle :: BAC Rectangle : ACq. In the same manner ABq : ABC :: ABC : BCq.* Q. E. D.] † Per 17. l. 6.

PROP. XXIII. Theorem.

EQuiangled Parallelograms (X, Z) have betwixt themselves a Proportion that is compounded of the Proportions of their Sides (AC to CB, and LC to CF.) Fig. 42.

That is, if you make CB to be to O, as LC to CF, X is to Z, as AC is to O.

Let IL, SB, meet together in Q. The Parallelogram X (*a*) is to the Parallelogram R, as AC is to CB; and R is (*b*) to Z, as LC is to CF; that is, as CB is to O. Therefore *ex æquo* X is to Z, as AC is to O. *Q. E. D.*

(a) Per 1. l. 6.
(b) By the same.

Corollaries.

FRom hence, and from 34. l. 1. it is manifest, Fig. 42.

1. That Triangles which have one Angle (at C) equal, have that Proportion betwixt themselves, which is compounded of the Proportions of the right Lines AC to CB, and LC to CF. Which Lines contain the equal Angle.

2. That Rectangles, and consequently all Parallelograms whatsoever, have betwixt themselves that Proportion which is compounded of the Proportions of the Base to the Base, and the Height to the Height. And in the same manner we reason about Triangles.

Fig. 42. 3. Hence the Proportion of Triangles and Parallelograms may be readily learned. Let X and Z be the Parallelograms, and their Bases A C, C B, and C L, C F be their Heights. Let it be made (a) as the Altitude C L, is to the Altitude C F, so is one of the Bases C B, to O. The Parallelogram X is to the Parallelogram Z, as A C to O.

(a) Per 12. l. 6.

PROP. XXIV. Theorem.

Fig. 43. *IN every Parallelogram (as S F) the Parallelograms which are about the Diameter (A B), to wit, (C L, O I) are both like to the whole Parallelogram, and to each other.*

By 27. 1. the Angles C, S, and L, F, are equal. By the same, E is equal to I, that is, by the same, equal to A itself; but B is common both to the whole S F, and the Part C L. Therefore the whole S F, and the Part C L, are equiangular. It remains to be shew'd, that they have the Sides opposite to the equal Angles proportional.

Because in the Triangles B C E, B S A, C E is parallel to S A, B C (by *Corol.* 1. *pr.* 4. *l.* 6.) will be to C E, as B S to S A: And C E will be to E B (by the same *Coroll.*) as S A to A B. But because in the Triangles E L B, A F B also, E L is parallel to A F; E B (by the same *Coroll.*) will be to E L, as A B to A F. Therefore *ex æquo* C E is to E L, as S A to A F. Therefore (by *Defin.* 1. *l.* 6.) C L and the whole C F are like. In the same manner, I might shew O I to be like to the whole S F. Therefore (*per* 21. *l.* 6.) C L and O I are also like betwixt themselves. *Q. E. D.*

PROP. XXV. Problem.

Fig. 46. *TO change a given Polygon (A) into another like to a given one (B).*

Or to make a Polygon equal to a given one (A) and like to another given one (B).

Upon

Upon CF the Side of the Polygon B, a like one to which is required, (by 45. l. 1.) make a Rectangle Q equal to B. Then upon FI (by the same *Prop.*) make a Rectangle R equal to A. It is manifest that CF and FI do make one right Line. Betwixt CF and FI find a mean Proportional FL *(a)*. Upon this, *(p.* 18. *l.* 6.) make a Polygon like to the given one B, this must also be equal to the given one A. (a) Per 13. l. 6.

For seeing by the Construction, CF, FL, FI, are three Proportionals, the Polygon B is to the Polygon like to it which is made upon FL, as CF is to FI *(per* 20, *l.* 6. and *Defin*, 10. *l.* 5.); that is, *(per* 1. *l.* 6.) as Q is to R. Therefore also by changing, as the Polygon B is to Q, so is the Polygon FL to R. But by the Construction, the Polygon B is equal to Q. Therefore also the Polygon upon FL, which is like to B, is equal to R; that is, by the Construction to the given A. That therefore is done which was required.

PROP. XXVI. Theorem.

LIKE Parallelograms (BD, FN) having a common Angle (A) are about the same Diameter. Fig. 44.

Draw the right Lines AE, CE. If you deny that AEC is a common Diameter to the Parallelograms BD and FN; let another right Line AGC which cuts FE in G, be the Diameter of BD, and draw the Parallel GH. The Parallelograms FH, BD will be therefore about the common Diameter AGC, and consequently (by 24. *l.* 6.) will be like. Therefore *(per defin.* 1. *l.* 6.) as BA to AD, so is FA to AH. But also as BA to AD, so is FA to AN, seeing BD, FN are like by the Hypothesis. Therefore FA is to AH, as the same FA is to AN. Which is absurd.

PROP. XXVII, XXVIII, XXIX.

THese cause Trouble to, and perplex Beginners, and are scarce of any Use.

PROP. XXX. Problem.

Fig. 45. *To cut a given right Line (A B) so that the whole (A B) shall be to one Segment (A C) as the same Segment is to the Remainder (C B).*

That is, as Geometricians speak, to cut a Line in extreme and mean Proportion,

By 11. *l.* 2. so cut A B in C, that the Rectangle under A B, C B, may be equal to the Square of A C. I say the Thing is done.

For by the 17th of this Book, as A B is to A C, so is A C to C B.

The Force of this Section of a Line is admirable in the inscribing and comparing regular Bodies.

PROP. XXXI. Theorm.

Fig. 47. *If from the Sides of a rectangular Triangle (A C B) like Figures whatsoever be describ'd, that which is oppos'd to the right Angle, will be equal to the two others (L, R) taken together.*

Here *Prop.* 47. *l.* 1. is made universal.

From the right Angle C let the Perpendicular C O be let down. Because (*per Coroll.* 2. *p.* 8. *l.* 6.) A B, B C, B O, are three Proportionals, F shall be to the Figure R, which is like to it, as A B the first, to B O the third Proportional, (to wit, by 20. *l.* 6. and *Defin.* 10. *l.* 5.) Again, because (by the aforesaid Corollary) BA, A C, AO, are three Proportionals, the Figure F shall (by the foresaid *Prop.* and *Defin.*) be to L, which is like to it, as B A the first, to A O the third Proportional. Because therefore F is to R as A B is to B O; and the same F is to L, as A B to A O; F shall also be to R and L taken together, as A B is to B O, A O, taken together. But A B is equal to the two B O, A O. Therefore also F shall be equal to the two R and L. *Q. E. D.*

Coroll.

Coroll.

FRom this Propofition we can eafily find one rectilinear Figure, equal and like to any Number of rectilinear Figures whatfoever, by the fame Method, whereby *Prop.* 1. *Schol. pr.* 47. *l.* 1. one Square is found equal to any Number of given Squares whatfoever. Only in the Demonftration, let 31. *l.* 6. be cited inftead of 47. *l.* 1.

Coroll. (2.) *A Circle upon the Hypotenufe of a Rectangle Triangle, is equal to two Circles defcrib'd upon the Sides, for all Circles are like amongft themfelves; and are to one another as the Squares of their Diameters, by the fecond of the 12th Book.*

Coroll. (3.) *From hence we may derive that Quadrature of Lunets (or little Moons) which* Hippocrates *of* Chios *firft taught.* Fig. 54.

For let ABC be a rectangle Triangle; and BAC a Semicircle to the Diameter BC: BNA a Semicircle defcrib'd on the Diameter AB; AMC a Semicircle defcrib'd upon the Diameter AC. Thus therefore the Semicircle BAC is equal to the Semicircles BNA, and AMC together. If therefore you take away the two Spaces BA, AC, common on both Sides, there will be left the two Lunets BNA, AMC, bounded on both Sides, with circular Lines equal to the rectilinear Triangle BAC. And if the Line BA be equal to the Line AC, and you let fall a Perpendicular unto the Hypotenufe BC, the Triangle BAO will be equal to the Lunet BNA, and the Triangle COA equal to the Lunet CMA. Q. E. I.

PROP. XXXII.

T*HIS is hardly of any Ufe, and hath nothing remarkable in it.*

PROP.

PROP. XXXIII. Theorem.

Fig. 48. *IN the same or equal Circles, the Angles whether at the Centers (as A B C, FO D); or at the Circumference (as A R C, F S D) have that Proportion betwixt themselves, which the Arches (A K C, F G D) on which they stand have. Understand the same Thing of Sectors.*

As for the Angles at the Centre and the Sectors, it will be demonstrated altogether in the same manner, in which *Prop.* 1. of this Book it was demonstrated, that Triangles of the same Height are as their Bases: Only where *Prop.* 38. *l.* 1. is cited there, let *Prop.* 29. *l.* 3. be cited here.

And because the Angles R and S at the Circumference are Halves of the Angles A B C, F O D, at the Centre, that which hath been demonstrated of these will be manifest also of those.

Corollary.

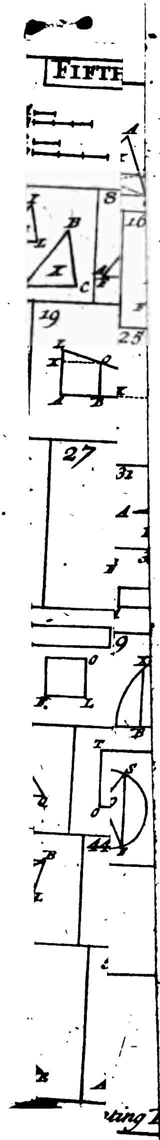
Fig. 49. 1. THE Angle (B A C) at the Centre, is to four right Angles, as the Arch B on which it stands, is to the whole Circumference.

For as B A C is to the right Angle B A F, so by this 33. the Arch B C is to the Quadrant B F. Therefore the Angle B A C is to four right Angles, as the Arch B C is to four Quadrants, that is, the whole Circumference.

2. The Arches I L, B C of unequal Circles, which do subtend equal Angles, whether at the Centre, as I A L and B A C, or at the Circumference, are like Arches.

For the Arch I L is (by *Coroll.* 1.) to its Circumference, as the Angle I A L, that is, B A C is to four right Angles; and the Arch B C is to its Circumference (by the same *Corollary*) as the same Angle B A C is to four right ones. Therefore I L is to its Circumference, as

B C



... Angle B A C is to four
t ones. Therefore I L is to its Circumference, as
B C

B C is to its. Therefore (by *Defin.* 4. *l.* 6.) the Arches I L and B C are like.

3. The Semidiameters (A B, A C) do take away from concentrical Circumferences like Arches I L, B C. This is manifest from *Coroll.* 2.

4. The Segments (B K C, I O L) which contain equal Angles (K, O) are like.

For by *Coroll.* 2. the Arches B C, I L, and consequently the Angles B K C, I O L, are like.

The

The Elements of Euclid.

BOOK XI.

With Us the Seventh.

TO the six first Books *Euclid* subjoins the Elements of Numbers, comprehended in the three following, the Seventh, Eighth, and Ninth; to which he also adjoins a Tenth, concerning incommensurable Quantities. We pass immediately from Planes to Solids; purposing to treat of Numbers separately: Seeing it will, I suppose, be more commodious for Learners, if the Elements of Geometry be not interrupted, by treating of any other Matter, but be had all together. Neverthelesss, when we shall cite the Propositions of this and the following Book, we shall not call these Books the Seventh, and the Eighth, but the Eleventh and the Twelfth, lest if we should depart from the every where receiv'd Order of *Euclid*, the Citation of Propositions should thereby be render'd more intricate.

This Book in a sort contains two Parts: In the first are laid the Foundation on which the whole Doctrine of the Solids, that is, of Bodies, depends. In the other the Affections of Parallelepipeds are propounded.

This Eleventh Book of Elements sets forth the first Principles of Solids. Nor can indeed the Properties of Bodies be known without it; and if we set upon almost any Part of the Mathematicks, without the knowledge of Solids, we shall labour in vain, or be at least at a great Loss. For the Spherical Doctrine of Theodosius, *Spherical Trigonometry also, a great Part of practical Geometry, Staticks, and Geography, depend upon it; and what Things occur of any great Difficulty in the Art*

Art of Dialling, in the Conic Sections, Astronomy, Dioptricks or Opticks, do all become more easy, the Principles of Solids being once understood. So that those who have deliver'd the Elements of Geometry, leaving out and setting aside this and the following Book, are to be reckon'd to have delivered the same very imperfectly.

Definitions.

1. A Solid or Body is that which hath Length, Breadth, and Thickness.

2. The Extreme of a Solid is a Surface.

3. The right Line (A B) is to the Plane (C C) right or perpendicular, when it makes right Angles (B A C, B A C) with all the right Lines (C A) in the Plane (C C) by which it is touch'd. *Fig.* 1. *l.* 11.

4. A Plane is right or perpendicular to a Plane, when all the right Lines (L Q) which are drawn in one of the Planes perpendicular to the common Section (X R) are right or perpendicular to the other Plane (A B C O). *Fig.* 2.

5. If the right Line (O L) stands upon a Plane not at right Angles, and from its highest Point (L) there be drawn to the Plane the Perpendicular (L P), and (O P) be join'd; the Angle (L O P) is said to be the Inclination of the Line (O L) to the Plane. *Fig.* 3.

6. If the Plane (R E) doth not stand perpendicularly upon the Plane (L Q), the Inclination of one to the other is the acute Angle (A B C), which is contain'd by the right Lines (A B and B C), which are drawn in both Planes perpendicular to the common Section (O E). *Fig.* 4.

7. A Plane is said to be alike inclin'd to a Plane, as is some other Plane to another, when the said Angles of their Inclinations are equal.

8. Parallel Planes, are those which being continued every way, are always distant from each other by equal Intervals.

9. Like solid Rectilinear Figures are those which are contain'd under like Planes, in Number equal.

10. A solid right-lin'd Angle is that which is contain'd under plain Angles more than two (B A C, C A O, O A B) which are not in the same Plane, meeting together in one Point. *Fig.* 5.

11. Equal

11. Equal solid Angles are those, which being conceiv'd to be put each within the other, do agree or perfectly coincide.

Like as a plain Angle is a mutual Inclination of Lines, so a solid Angle is an Inclination of Surfaces.

Fig. 6, 7, 8. 12. A Prism is a solid Figure, comprehended by Planes; amongst which two opposite ones (OFE, ACB) are parallel, equal and like.

Fig. 8. 13. A Parallelepiped is a Solid contain'd under quadrilateral Planes, of which the Opposites are parallel.

14. If six Planes in which the Opposites are parallel be Squares, the Solid contain'd by them will be a Cube.

PROPOSITION I. Theorem.

Fig. 9. *ONE Part (A C) of a right Line cannot be in a Plane (O E); and another part (C B) out of it.*

It is clear of it self, from the Definition of a Plane and a right Line. See *Defin.* 4. and 7. *l.* 1.

PROP. II. Theorem.

Fig. 10. *EVery Triangle is in one Plane: And two right Lines cutting each other, are in the same Plane.*

For if a Plane be applied to one of its Sides, and to the Point of meeting of the other two, it will be evident that the whole Triangle is in that Plane.

PROP. III. Theorem.

Fig. 11. *IF two Planes (AB, CD) cut each other, (E F) their common Section is a right Line.*

It is manifest from the Definition of a Plane.

But

But we may demonstrate it thus. If E F the common Section be not a right Line, let there be drawn in the Plane C D the right Line E O F, and in the Plane A B the right Line E Q F. The two right Lines therefore E O F, E Q F, will include a Space. Which is absurd.

PROP. IV. Theorem.

If a right Line (B A) be perpendicular to two right Lines (C A X, F A S) which cut each other, it will also be perpendicular to the Plane which is drawn thro' them. Fig. 12.

If you deny it, let another right Line B Q be perpendicular to the Plane of the right Lines A C, A F. Join A Q, and to this in the Plane F A C draw the Perpendicular Q O. This being produced, will necessarily cut (as is gather'd from *Schol. Prop.* 31. *l.* 1.) one of the right Lines C A X, F A S, or both, wheresoever the Point Q shall be. Therefore let it cut C A X in O, and let B O be join'd. Because therefore the Angle B A O is by the Hypothesis a right one;

The Square of B O shall be equal to

$$\left.\begin{matrix} \text{B A Squ.} \\ + \\ \text{A O Squ.} \end{matrix}\right\} (a).$$

(a) Per 47. l. 1.

But because B Q is suppos'd perpendicular to the Plane F A C, and consequently (by *Defin.* 3. *l.* 11.) makes a right Angle with A Q;

$$\text{B A Squ. is equal to} \quad \left.\begin{matrix} \text{B Q Squ.} \\ + \\ \text{A Q} \end{matrix}\right\} (b)$$

(b) Per 47. l. 1.

And because the Angle A Q O is by the Construction a right one;

$$\text{A O Squ. is equal to} \quad \left.\begin{matrix} \text{O Q Squ.} \\ + \\ \text{A Q Squ.} \end{matrix}\right\} (c).$$

(c) By the same.

$$\text{Therefore B O Squ. is equal to} \quad \left.\begin{matrix} \text{B Q Squ.} + \\ \text{O Q Squ.} + \\ \text{A Q Squ. twice taken.} \end{matrix}\right\}$$

There-

Therefore BO Square is greater than the Squares of BQ and OQ; and (as is clear from *Prop.* 47. *l.* 1.) consequently BQO is not a right Angle. Therefore BQ is not perpendicular to the Plane (by *Defin.* 3. *l.* 11.) CAF. Therefore the Proposition is manifest.

Scholium.

FRom its being suppos'd that BQ is perpendicular to the Plane FAC; it is directly demonstrated that it is not perpendicular to that Plane; and consequently from the denial of the Assertion of the Theorem, the same Assertion is directly proved. This Demonstration, as to the Substance of it, is *John Cierman*'s.

PROP. V. Theorem.

Fig. 13. *IF three right Lines (BA, CA, FA) be perpendicular to the same right Line (AR) at the same Point (A); those three will be in one Plane.*

For, if it may be, let one of them BA be in another Plane (RO) which may cut LQ the Plane of the other two CA, FA, in the right Line AO. Because by the Hypothesis RA stands perpendicularly upon the two CA, FA, it will be perpendicular to the Plane LQ (by the foregoing). Therefore RA makes a right Angle with AO (by *Defin.* 3. *l.* 11.) But also by the Hypothesis RAB is a right Angle. Therefore the Angles RAB and RAO are equal. Which is absurd.

PROP. VI. Theorem.

Fig. 14. *RIght Lines (AB, CD) which are perpendicular to the same Plane (CF) are parallel.*

It might be taken for granted as a Thing of it self known; but we may demonstrate it thus.

BD being join'd, make in the Plane FE the Line DG perpendicular to BD, and equal to BA; and let DA,

DA, GA, GB, be join'd. The right Lines BD, DG, are equal to BD (*a*) and BA; and the Angles BDG, (*b*) DBA are right ones. Therefore (*per*. 4. *l*. 1.) AD, BG, are equal. Therefore the Triangles ABG, GDA are equilateral to each other, and consequently the Angles ABG, ADG are equal. But ABG (by *Defin*. 3. *l*. 11.) is a right Angle. Wherefore ADG is also a right one. But BDG also by the Construction, and CDG by *Defin*. 3. are right Angles. Therefore GD is perpendicular to three Lines CD, AD, BD. Therefore CD is (*c*) in one Plane with AD, and BD. But AB also is in one Plane (*per* 2. *l*. 11.) with AD and BD. Therefore AB, CD are in one Plane. Therefore seeing the Angles ABD, CDB (by *Defin*. 3. *l*. 11.) are right ones, AB, CD will (*per* 29. *l*. 1. and *Defin*. 36. *l*. 1.) be parallel Lines. *Q. E. D.*

(a) By the Construction

(b) *Per Def.* 3. *l*. 11.

(c) By the foregoing.

PROP. VII. Theorem.

Fig. 15.

A Right Line (EF) cutting right Lines (AB, CD) placed in the same Plane, is in one and the same Plane with them.

It might be taken for granted. But he that will may thus demonstrate it.

Let another Plane cut the Plane of the right Lines AB, CD, in the Points EF, If now EF is not in the Plane of AB, CD, it is not the common Section. Let EGF therefore be so. Therefore (*per* 3. *l*. 11.) EGF is a right Line; the two right Lines therefore EF, EGF inclose a Space. Which is absurd.

Corollary.

HEnce it follows, that if EF cut the Parallels AB CD, it is in the same Plane with them. For (by *Defin*. 36. *l*. 1.) any two Parallels are in the same Plane.

PROP. VIII. Theorem.

Fig. 14. *IF of two Parallels (AB, CD) one (AB) be perpendicular to a Plane (EF); the other also (CD) will be perpendicular to the same Plane.*

It might be taken for granted. If the Demonstration be requir'd, it is as follows.

[*BD, AD being drawn; in the Plane EF make GD perpendicular to BD. It will also (see the Demonstration of* Prop. 6. l. 11.) *be perpendicular to AD. Therefore* (per 4. l. 11.) *GD will be perpendicular to the Plane ABD, that is (by the foregoing* Coroll.) *to the Plane CBDA. Wherefore* (per Def. 3. l. 11.) *CDG is a right Angle. But the Angle CDB is also a right one; forasmuch as with ABD which* (per Defin. 3. l. 11.) *is a right Angle, it maketh two right ones* (per 27. l. 1.) *Therefore* (per 4. l. 11.) *CD is perpendicular to the Plane GDB or EF.* Q. E. D.]

PROP. IX. Theorem.

Fig. 16. *RIGHT Lines (AB, EF) which are parallel to the same right Line (CD) altho they be not in the same Plane with it, are also parallel betwixt themselves.*

Altho it might be taken for granted, yet we will demonstrate it thus.

In the Plane of the Parallels AB, CD, draw GK perpendicular to CD. Likewise in the Plane of the Parallels EF, CD, draw HK perpendicular to CD. Therefore, (*a*) CK is perpendicular to the Plane GKH. Therefore, seeing AG, EH, are parallel to CK, the same AG, EH (*b*) will be perpendicular to the Plane GKH. Therefore AG, EH (*c*) are parallel. *Q. E. D.*

(a) Per 4. l. 11.
(b) Per 8. l. 11.
(c) Per 6. l. 11.

PROP.

PROP. X. Theorem.

If two right Lines (AC, BC) be parallel to two right ones (DF, EF); albeit they be not in the same Plane, they comprehend equal Angles (C and F). Fig. 17.

Let CA, CB, be made equal to FD, FE, and let DE, AB, DA, FC, EB be drawn. Seeing AC, FD are parallel and equal, AD also and CF will (a) be parallel and equal. In like manner I might shew BE, CF to be parallel and equal. Therefore AD, BE, are also parallel (b) and equal (*per Axiom.* 1.) Therefore (*per* 33. *l.* 1.) AB, DE, are equal. Seeing therefore the Triangles BAC, EDF are equilateral to each other, the Angles C and F (c) are equal. *Q. E. D.*

(a) *Per* 33. *l.* 1.
(b) By the foregoing.
(c) *Per* 8. *l.* 1.

PROP. XI. Problem.

To draw a Perpendicular to a given Plane (AB) from a Point given without it (C.) Fig. 18.

The Construction. In the Plane AB draw any right Line as DF, unto which, from C erect the Perpendicular CE. Then in the Plane AB thro' E draw AEM perpendicular to the same DF. Then to AM from C draw the Perpendicular CG. I say that CG is perpendicular to the Plane, AB.

Thro' G let HG be drawn parallel to DF. By the Construction DE is perpendicular to GE and EM. Therefore DE is perpendicular to the Plane EM (d), as also is HG (e). Therefore (by *Defin.* 3. *l.* 11.) CG is perpendicular to HG. But CG by the Construction is also perpendicular to EM. Therefore (f) CG is perpendicular to the Plane AB. Which was the Thing propos'd.

(d) *Per* 4. *l.* 11.
(e) *Per* 8. *l.* 11.
(f) *Per* 4. *l.* 11.

[Scholium. *In Practice thus: Let there be a Cord or Rule fastned to the given Point A: And from the same, let there be describ'd by the end of it B in the Plane given the Circle BCFL. The Line AK which connects the given Point and the Centre of the Circle, will be perpendicular to the given Plane.*] Fig. 20. l. 12.

PROP. XII. Problem.

Fig. 19. *From a given Point (A) in any Plane (E F) to erect a Line perpendicular to the same Plane.*

From any Point D without the Plane E F make D B (by the foregoing) perpendicular to the Plane E F. And B A being join'd, draw A C parallel to D B. I say the thing is done. The Demonstration is manifest from *Prop.* 8.

Corollary.

IN Practice, from the given Point a Perpendicular is erected to the given Plane, if a Square O K N be applied to the given Point [and be turn'd round.]

PROP XIII. Theorem.

Fig. 20. *Lines drawn from the same Point cannot be both perpendicular to the same Plane (A B).*

For if they were, they wou'd (by *Prop.* 6.) be parallel. Which cannot be.

PROP. XIV. Theorem.

Fig. 21. *IF the same right Line (A B) be perpendicular to two Planes (F G, L Q); the Planes will be parallel.*

Let there be taken in either of the Planes as F G, any Point C, from which let C E be drawn parallel to A B, and meeting the Plane L Q in E. Then C E (*per.* 8. *l.* 11.) will be perpendicular to both Planes F G, L Q. Wherefore if A C, B E, be join'd, the Angles A, B, (by *Def.* 3. *l.* 11.) will be right ones. Therefore (*per* 29. *l.* 1.) A C, B E, are parallel. Therefore A C E B is a Parallelogram; and consequently C E, which hath been already

dy shewn to be perpendicular to both Planes, is equal (*per* 34. *l.* 1.) to A B. In the same manner I might shew that all the Perpendiculars to both Planes are equal. Eherefore (by *Defin.* 8. *l.* 11.) the Planes are parallel. *Q. E. D.*

PROP. XV. Theorem.

IF two right Lines (B A, C A) touching each other be parallel to two other right Lines which also touch one another (E D, F D); the Planes likewise which are drawn thro' them, will be parallel. Fig. 22.

From A let there be drawn A G perpendicular to the Plane E F, and let G H, G I, be parallel to D E, D F. These (*per* 9. *l.* 11.) will also be parallel to A C, A B. Seeing therefore the Angles I G A, H G A, be (by *Defin.* 3. *l.* 11.) right; C A G, B A G, will also (*a*) be right Angles. Therefore G A which is perpendicular to the Plane E F, will also be perpendicular to the Plane B C (*b*). Therefore the Planes B C, E F, are (by the foregoing) parallel. *Q. E. D.*

(a) *Per* 27. *l.* 1.

(b) *Per* 4. *l.* 11.

PROP. XVI. Theorem.

A Plane (E H F G) cutting parallel Planes (A B, C D) makes the Sections in them (E H, G F) parallel. Fig. 23.

If not, seeing they be in the same intersecting Plane, they will meet somewhere (by *Schol. Prop.* 21 *l.* 1.) as in I. Wherefore seeing the whole Lines H E I, F G I be in the Planes * A B, C D produc'd, these Planes also will meet in I. Which is absurd, and contrary to *Defin.* 8. *l.* 11.

* *Per* 1. *l.* 11.

PROP. XVII. Theorem.

Fig. 24. *PArallel Planes cut right Lines (B D and G H) proportionally.*

Let the right Lines B H, G D be drawn in the Planes P V, T Q; and likewise let B G be drawn meeting the Plane R S in F, and let F C, F I be join'd. The Plane of the Triangle B G D cutting parallel Planes, makes the Sections C F, D G, parallel (by the foregoing). (a) Per 2. l. 6. Therefore B C is to C D, as B F (*a*) to F G. Again, the Triangle B H G cutting parallel Planes makes the Sections (by the foregoing) B H, F I parallel. Therefore (b) Per 2. l. 6. H I is to I G as (*b*) B F to F G; that is, (as I have already shew'd) as B C is to C D. *Q. E. D.*

PROP. XVIII. Theorem.

Fig. 25. *IF a right Line (F E) be perpendicular to a Plane (A B); all the Planes which are drawn thro' it are perpendicular to the same Plane (A B).*

Let the Plane G C be drawn thro' F E, making C D the common Section with A B; and let the Lines H K be drawn in the Plane G C, perpendicular to the common Section C D. Now seeing by the Construction H K is perpendicular to the same common Section to which F E is perpendicular by the Hypothesis, K H and F E must be parallel (by 29. *l.* 1.) Therefore H K is also perpendicular to the Plane A B (*per* 8. *l.* 11.) Therefore the Plane G C is perpendicular to the Plane A B (*per Defin.* 4. *l.* 11.)

PROP. XIX. Theorem.

Fig. 26. *IF two Planes (M F, G D) cutting each other be both perpendicular to the same Plane (A B); their common Section also will be perpendicular to that Plane (A B).*

For

For seeing by the Hypothesis the Plane MF is perpendicular to the Plane AB; it is manifest by Definition 4. that there may be drawn in the Plane MF from the Point L a Perpendicular to the Plane AB. Again, by the Hypothesis GD is perpendicular to that Plane AB, and therefore there may be also drawn in the Plane GD from the Point L a Perpendicular to the Plane AB. But from the Point L (*a*) there can be erected only one Perpendicular to the same Plane AB. Therefore the Perpendicular to the Plane AB, which is drawn from the Point L, must be found in both the Planes MF and GD, and consequently LK, the common Section of those two Planes MF and GD, is perpendicular to the Plane AB. *Q.E.D.* (a) Per 13. l. 11.

PROP. XX. Theorem.

If a solid Angle (A) is contain'd under three plain Angles (BAC, CAD, DAB); any two of these is greater than the third. Fig. 27.

If the three Angles be equal, the Assertion is manifest at first Sight; and it is as certain, if they be unequal. For let BAD be the greatest; and from BAD cut off BAE equal to BAC, and make the Line AC equal to AE. Thro' E let there be drawn a right Line meeting AB and AD in B and D, and let BC, DC be join'd. Because (by the Construction) the Angles BAE, BAC are equal, as likewise the Sides BA, AE, equal to the Sides BA, AC, the Bases also BE, BC, will be equal (*b*). And because BC, CD (*c*) are greater than BD, the Equals BE, BC being taken away, there remains CD greater than ED. But the Sides EA, AD, are (by the Construction) equal to the Sides CA, AD. Therefore the Angle (*d*) CAD is greater than the Angle EAD. Seeing therefore the Angle BAC is equal by the Construction to the Angle BAE, those two Angles together BAC, CAD are greater than the whole BAD. *Q. E D.* (b) Per 4. l. 1. (c) Per 20. l. 1. (d) Per 25. l. 1.

PROP. XXI. Theorem.

THE plain Angles constituting any solid Angle whatsoever, are less than four right ones.

Fig. 281. Let A be a solid Angle; let the right Lines BC, CD, DE, EF, FB, be subtended to the Plain Angles which make up the solid one, so that those right Lines be all in one Plane. Which being done, there is constituted a Pyramid, whose Base is the Polygon BCDEF; A is the Top, and it is contain'd under so many Triangles G, H, I, K, L, as there are plain Angles which compose the solid Angle A. And now because the two Angles ABF, ABC, are (by the foregoing) greater than the third FBC; and the two ACB, ACD, are greater than the third BCD, and so on: All the Angles of the Triangles G, H, I, K, L, about the Base, as taken together, are greater than all the Angles of the Base B, C, D, E, F, taken together. But the Angles of the Base together with four right ones, make twice so many right Angles (by *Theorem* 2, *Schol.* after 32. *l.* 1.) as there are Sides, or, which is the same, as there are Triangles. Therefore all the Angles of the Triangles about the Base, together with four right ones, make more than twice so many right Angles as there are Triangles. But the same Angles about the Base, together with the Angles that compose the Solid, make up *(a)* twice so many right Angles as are the Triangles. It is manifest therefore, that the Angles which compose the solid Angle A are less than four right ones. *Q. E. D.*

(a) Per 32. l. 1.

Corollary.

FRom this and the foregoing it is obvious to collect, that a solid Angle may be compos'd of any three plain Angles, which are less than four right ones, if so be that any two of them be greater than the other.

Scho-

Scholium.

From this Propoſition is demonſtrated that famous Theorem, That only three regular and equal plain Figures can contain a Body; to wit, equilateral Triangles, either 4, or 8, or 20; 6 Squares, and 12 Pentagons: And conſequently that there are only five regular Bodies. A Pyramid which is contain'd under 4; an Octaedrum which is comprehended by 8; and an Icoſaedrum, which is bounded by 20 equilateral Triangles; a Cube which is contain'd under 6 Squares; and the Dodecaedrum under 12 regular and equal Pentagons. Now a Body is called Regular which is comprehended under regular and equal Planes.

Demonſt. A ſolid Angle cannot be compos'd of only two equilateral Triangles; three at leaſt are requir'd.

Of three equilateral Triangles meeting in one Point, there may be conſtituted the ſolid Angle of a Pyramid; of four, the ſolid Angle of an Octaedrum; of five, the ſolid Angle of an Icoſaedrum: Foraſmuch as both 3, 4, and 5 Angles of an equilateral Triangle are leſs than 4 right ones, as is gathered from *Coroll.* 12. *Prop.* 32. *l.* 1.

And becauſe three Angles of a regular Pentagon (as is gathered from *Coroll. Prop.* 11. *l.* 4.) are leſs than four right ones, three Pentagons meeting in one Point will conſtitute a ſolid Angle; that of the Dodecaedrum.

That of three Squares meeting in one Point may be compos'd the ſolid Angle of a Cube, is manifeſt of it ſelf. And thus there ariſe five regular Bodies.

But that there are no more than theſe five, is thus proved.

Six Angles of an equilateral Triangle make juſt four right ones. For one is two Thirds of one right one; and therefore ſix ſuch will make (by *Coroll.* 12. *Prop.* 32. *l.* 1.) twelve Thirds of one right one, that is, four right ones. And therefore of ſix equilateral Triangles a ſolid Angle cannot be compos'd; much leſs of more.

That of four Squares a ſolid Angle cannot be made, much leſs of more, is manifeſt in it ſelf.

Four Angles of a regular Pentagon are greater than 4 right ones. For (by *Coroll. Prop.* 11. *l.* 4.) each of them makes ſix Fifths of one right one, Therefore a ſolid

Angle cannot be made of four such Pentagons; much less of more.

Nor can a solid Angle be compos'd of any other regular Figures whatsoever. Three Angles of a regular Hexagon (by *Coroll.* 2. *Prop.* 15. *l.* 4.) are equal to four right ones. For one makes four Thirds of one right one; and therefore three make twelve Thirds of one, that is, four entire right ones. Therefore of three Hexagons a solid Angle cannot be made up; much less of more.

But seeing three Angles of a regular Hexagon are equal to four right ones, three Angles of any other Figures whatsoever greater than an Hexagon, as of an Heptagon, Octagon, &c. will be greater than four right ones. Wherefore it is manifest that the rest of the regular Figures are all incapable of composing a solid Angle; and consequently that there can be no regular Bodies besides the five foregoing.

PROP. XXII, XXIII.

ARE very prolix, and tedious to Beginners, and scarce at any time come into Use.

PROP. XXIV. Theorem.

Fig. 29. *THE Planes which contain a Parallelepiped are (1.) Parallelograms. (2.) The opposite ones are Simliar; and (3.) equal.*

Part I. The Plane A F cutting the Planes B D, F H, which by *Defin.* 13. are parallel, makes (*a*) the Sections B A, F E, parallel. Again, the Plane A F cutting the Planes A H, B G, which by the same Definition are parallel, (by the same) makes the Section A E, B F, parallel. Therefore B A E F is a Parallelogram. By the like Argument the rest of the Planes of the Parallelepiped may be prov'd to be Parallelograms.

(a) Per 16. l. 11.

Part II. Because it is manifest from the first Part, that A B, B C, are parallel to E F, F G; the Angles A B C, E F G, must be (*b*) equal. Wherefore seeing the alternate Sides also are equal, the opposite Parallelograms

(b) Per 10. l. 11.

grams

grams B D, F H, are like or similar. And the same of the rest.

Part III. This is manifest from the first Part, and 4th, or 8th of the First Book.

PROP. XXV. Theorem.

If a Parallelepiped (G F D I) or any Prism whatever be cut by a Plane (N P) that is parallel to the opposite Sides; there will be this Proportion, as the Base (D C P O) is to the Base (O P F E) so is the solid (G P) to the solid (N F). Fig. 30.

This is demonstrated in the same manner as *p.* 1. *l.* 6.

Corollary.

A Prism cut by a Plane parallel to the opposite Planes, hath a Section like, and equal to the opposite Planes.

PROP. XXVI, XXVII.

Are not necessary.

PROP. XXVIII. Theorem.

A Plane passing thro' the Diameters of opposite Planes (A C, E G) cuts the Parallelepiped into two equal Prisms. Fig. 29.

Because (*a*) B G, B E, are Parallelograms; C G, A E, are equi-distant from the same B F. Therefore (*b*) they are also parallel betwixt themselves, and consequently are in one Plane. Therefore the right Lines A C, E G, are (*c*) in one Plane. But now that a Plane drawn thro' them doth cut the Parallelepiped into two equal Prisms, is thus shew'd. Let the Prism A E G C D H be understood

(a) Per 24. l. 11.
(b) Per 9. l. 11.
(c) Per 7. l. 11.

ſtood to be ſo conſtituted upon its Plane AECG, that the Angles D, H, bend towards the Angles B, F. It is manifeſt that it will ſtill be betwixt the parallel Planes BADC, FEHG. But then D muſt needs fall upon B, and H upon F. For let D fall without B, if it may be, in N. The Angle BAC (*a*) is equal to the Angle DCA. But DCA is equal to NAC (for it is one and the ſame Angle.) Therefore BAC and NAC are equal: Which is abſurd. Therefore D falls upon B; and for the ſame Cauſe H upon F. Therefore the Priſm AEGCDH coincides with the Priſm ACGEFB. and conſequently they are equal (by *Axiom* 7.)

(a) Per 27. l. 1.

PROP. XXIX, XXX. Theorems.

Fig: 31.

THE Parallelepipeds (FEAGKIMC) and (FEBHLOMI) which have the ſame Baſe (EFIM) and the ſame Altitude, and conſequently exiſt between parallel Planes (EFIM) and (GAOL) are equal.

For they either exiſt betwixt the lateral parallel Planes EAOM and FGLI, or not. Let the firſt be ſuppos'd. From the 24th of this, and the 8th of the firſt Book, it is manifeſt that the Triangles AEB, CMO, likewiſe GFH, KIL, are equilateral and equiangular to each other. Wherefore, as in the foregoing, I might ſhew that the Priſms CMOLIK, and AEBHFG, being laid upon each other will coincide, and conſequently are equal. Wherefore the common ſolid FEBHKCMI being added, the whole Parallelepipeds FEAGKIMC and FEBHLOMI are equal. *Q. E. D.*

Then let the Parallelepiped FXQEMIPR not exiſt betwixt the ſame lateral parallel Planes with the Parallelepiped FEAGKCMI. Here, becauſe by the Hypotheſis, GK, AC, RP, QX, are in one Plane, which is parallel to the Baſe EFIM; let RP, QX, cut GK in L and H, and AC in O and B; and let EB, MO, FH, IL, be join'd. It is eaſy now to ſhew that the Planes containing the Solid FEBHLOMI are Parallelograms, the oppoſite ones of which are equi-diſtant, and conſequently that that Solid is (by *Defin.* 13. *l.* 11.)

a Parallelepiped. But to this by the first Part the Parallelepipeds FXQEMIPR, and FEAGKCMI, are each of them equal. Therefore they are also equal betwixt themselves.

Corollary.

THIS Proposition is like to the 35th of the first Book; for it affirms concerning Solids, what that doth touching Planes. Wherefore the Demonstration of the rest of the Cases will be like also.

PROP. XXXI. Theorem.

PArallelepipeds upon equal Bases (AO and EG) and in the same Altitude (S) are equal. Fig. 33.

First, let the Parallelepipeds have their Sides perpendicular to the Bases. Unto the side FG produc'd let there be made a Parallelogram GMKH equal and like to the Parallelogram AO; and the Parallelogram GMPR being perfected, let the right Lines PM, RG meet KH in Q and L. And now let Parallelepipeds be understood to be constituted upon GK, GQ, GP, whose Sides are perpendicular to the Bases, and S is their common Altitude. The Solid EGS is to the Solid GPS, as EG (*per* 25. *l.* 11.) is to GP; that is, (because EG, AO, are equal by the Hypothesis) as AO to GP; that is, by the Construction, as GK is to GP; that is, as GQ is to GP (*per* 35. *l.* 1.); that is, as the Solid GQS is to the same Solid GPS. (*per* 25. *l.* 11.) Because therefore the Solids EGS and GQS have the same Proportion to the Solid GPS, the Solid EGS will be equal to the Solid GQS; that is, to the Solid GKS (*per* 29. *l.* 11.) that is, (because the Bases GK, AO, are equal and like by the Construction) to the Solid AOS, as it appears from 29. *l.* 11. and even in it self. Which was the thing propos'd. Note, that in this reasoning, the Solids are suppos'd to be right or perpendicular ones.

Then let the given Parallelepipeds EGS, AOS have their Sides oblique to the Bases EG, AO. Let there now

now be made upon E G, A O, Parallelepipeds, whose Sides are perpendicular to the Bases in the Height S; these will be equal to the oblique ones by 29th or 30th. Wherefore seeing by the first Part, right Parallelepipeds are equal betwixt themselves, the oblique ones will be equal betwixt themselves likewise. *Q. E. D.*

PROP. XXXII. Theorem.

Fig. 34. *ALL Parallelepipeds whatever of equal Height, are betwixt themselves as their Bases.*

Let G O and A be the Bases. Upon C O make the Parallelogram O E equal to A.

Upon B C, O E, let Parallelepipeds be understood to be erected in the Altitude K; these therefore will be Parts of one Parallelepiped B E K. Therefore the Parallelepiped O E K, is to the Parallelepiped B C K, as the Base O E, to the Base B C (*per* 25. *l.* 11.); that is, by the Construction, as the Base A is to the Base B C. But because the Bases O E and A are equal, the Parallelepipeds O E K and A K are equal (by the foregoing). Therefore also the Parallelepiped A K is to the Parallelepiped B C K, as the Base A is to the Base B C. *Q. E. D.*

Scholium.

THAT which hath here been shew'd of Parallelepipeds, will be demonstrated in the Twelfth Book of Pyramids, *Prop.* 6. Of all Prisms whatever, in *Coroll.* 1. after *Prop.* 9. Of Cones and Cylinders, *Prop.* 11.

PROP. XXXIII. Theorem.

Fig. 35. *LIKE Parallelepipeds (H A and C M) are in a triplicate Proportion of their homologous Sides (A B, B C).*

Let the Parallelepipeds A H, C M, be like. Therefore all their Planes (by *Defin.* 9. *l.* 11.) are like; and conse-

consequently A E (by *Defin.* 1. *l.* 6.) is to B C, as E B to B O; and as F B is to B G, so is E B to B O. Moreover the Angles of the Planes are also equal (by the same). Therefore let the Solids A H, C M, be so plac'd, that the equal Angles C B O, A B E, may be opposite, and the Sides A B, C B, may lie so as to make one streight Line; and then E B, O B will also lie so as to make one streight Line. Now let Solids be imagin'd to be constituted upon the Planes B Q and B C, in such sort that the Solids K B, H A, may be one Parallelepiped, and K B, P O, may make one Parallelepiped, and P O, C M, may make one Parallelepiped likewise. The Solid H A is to the Solid K B (*per* 25. *l.* 11.) as A E to B R; that is, (*per* 1. *l.* 6.) as A B to B C; that is, (as I shew'd above by the Hypothesis) as E B is to B O; that is, (by the same) as E C is to B Q; that is, (*per* 25. *l.* 11.) as the same Solid K B is to the Solid P O. Therefore the three Solids H A, K B, P O, continue the same Proportion. But now the Solid K B is to the Solid P O (by the same) as the Base B R is to the Base B Q; that is, (*per* 1. *l.* 6.) as E B is to B O; that is, as F B is to B G, as it was shew'd above by the Hypothesis; that is, (by the same) as the Plane F C is to the Plane B S; that is, (*per* 25. *l.* 11.) as the same Solid P O again is to the Solid C M. Therefore the four Solids, H A, K B, P O, C M, are continually proportional. Therefore (by *Defin.* 10. *l.* 5.) the Proportion of the first H A to the fourth C M is triplicate of the Proportion of the first H A to the second K B; that is, triplicate to the Proportion (*per* 25. *l.* 11.) of A E to B R; that is, triplicate (*per* 1. *l.* 6.) to the Proportion of the homologous Sides, A B to B C. *Q. E. D.*

[Coroll. (1.) *Hence if there be four right Lines continually proportional; as is the first to the fourth, so is a Parallelepiped describ'd upon the first, to a Parallelepiped like, and in like manner describ'd upon the second.*

(2.) *Upon this also depends that most famous Problem concerning doubling the Cube; of which afterwards, Schol. p.* 18. *l.* 12.

(3.) *Hence also is to be corrected the Error of those, who suppose that the Proportion of like Solids is the same as is that of their Sides. For the Cube of a Line, which is double to another Line, is not only double to the*

3 *the*

the other, but as 8 to one. And the Cube of a Line, which is triple to another Line, is not only triple to the other Cube, but contains it 27 Times. For 1 : 2 : 4 : 8 ∸ *and* 1 : 3 : 9 : 27 ∸, *and the same Thing is to be said of all like Bodies whatsoever; as will appear afterwards.*

(4.) *Hence the triplicate Proportion of any Quantities whatsoever is the Proportion of the Cubes of the same Quantities. Let there be any two Quantities in the triplicate Proportion of the Quantities AB, BC; they shall also be as AB Cube is to BC Cube.*]

Scholium.

THAT which hath here been shew'd of Parallelepipeds will be demonstrated *Book* 12. Of Pyramids *Prop.* 8. Of all Prisms whatsoever, *Coroll.* 2. *Prop.* 9. Of Cones and Cylinders, *Prop.* 12. Of Spheres, *Prop.* 18.

PROP. XXXIV. Theorem.

Fig. 36. IF *the Parallelepipeds (BM, CK) be equal, their Bases and Altitudes are reciprocally proportional; (that is, the Base AM is to the Base FK, as reciprocally the Height FC is to the Height AB).*

And if they be reciprocally proportional, their Bases and Altitudes are equal.

Part I. First let the Sides be perpendicular to the Bases. If now the Altitudes of the Solids BM, CK be equal, the thing is manifest.

If the Altitudes be unequal, from the greater FC cut off FE equal to BA: and thro' E draw the Plane EL parallel to FK. The Base AM is to the Base FK, (*per* 25. *l.* 11.) as the Solid BM is to the Solid EK; that is, (because by the Hypothesis the Solids BM, CK are equal) as the Solid CK is to the Solid EK; that is, (by the same) as CG is to EG; that is, (*per* 1. *l.* 6.) as CF is to EF; that is, by the Construction, as CF to BA. *Q.E.D.*

Then let the Sides be oblique to the Bases. Let right Parallelepipeds be erected upon the same Bases in the same

same Height. The oblique Parallelepipeds will (*per* 29. *and* 30. *l.* 11.) be equal to these: Wherefore seeing these, by the first Part, have their Bases and Altitudes reciprocal, those also shall be so likewise. *Q. E. D.*

Part. II. Let the Altitudes be unequal, and the Sides perpendicular to the Bases; and from the greater CF take EF equal to AB. The Solid BM is to the Solid EK, (*per* 32. *l.* 11.) as AM is to FK, that is, by the Hypothesis, as CF is to AB; that is, by the Construction, as CF is to EF; that is, as CG is to (*a*) EG; that is, (*b*) as the Solid CK is to the same Solid EK. Therefore the Solids BM and CK have the same Proportion to EK: Therefore they are equal. *Q. E. D.*

(a) *Per* 1. *l.* 6.
(b) *Per* 25. *l.* 11.

Corollaries.

WHAT Affections have been demonstrated of Parallelepipeds, *Prop.* 29, 30, 31, 32, 33, 34, do also agree to triangular Prisms, which are the Halves of Parallelepipeds. As is manifest from *Prop.* 28. Therefore,

1. Triangular Prisms, which are of equal Height, are as their Bases, A, B.

2. If they be like, their Proportion is triplicate to the Proportion of the Sides, opposite to the Angles.

3. If they be equal, they reciprocate their Bases and Altitudes; and if they reciprocate their Bases and Altitudes, they are equal. *Fig.* 37.

Scholium.

WHAT hath here in *Prop.* 34. been shew'd of Parallelepipeds, will be demonstrated in the 12th Book of Pyramids, *Prop.* 9. Of all Prisms whatsoever, *Coroll.* 3. after *Prop.* 9. Of Cones and Cylinders, *Prop.* 15.

M PROP.

PROP. XXXV.

IS very long, and subservient to the following Proposition, which we will demonstrate without it.

PROP. XXXVI. Theorem.

Fig. 38. *A Parallelepiped (D H) made of three proportional Right Lines (A, B, C) is equal to a Parallelepiped (I N), which is made of the Mean (B), and is equiangular to the former.*

Let the Base F D of the Parallelepiped D H have the Side E F equal to A, and the other Side E D equal to C: And the Side E G which stands upon the Base equal to B. Thus the Parallelepiped D H will be made of the three right Lines, A, B, C. Then let the Three Sides, L X, I X, X M, (and consequently all the rest) of the Parallelepiped I N be equal to the middle Line B: And the solid Angle X equal to the solid Angle E; the Parallelepiped I N will be made of the Mean B; and be equiangular to the former. I say also that it is equal.

For seeing by the Hypothesis and the Construction, as F E is to L X, so reciprocally I X is to D E, the Bases * D F, I L will be equal. Now because the solid Angles at E and X are equal; if they be put within one another, † they will coincide; and because of the Equality of the right Lines, E G, X M, the Points M and G, will coincide. Wherefore both the Solids will have one perpendicular Altitude; to wit, the right Line which is let fall from the Points M, G, (now become one) unto the Plane of the Base. The Solids therefore D H, I N * are equal. *Q. E. D.*

* Per 14. l. 6.

† Per defin. 11. l. 11.

* Per 32. l. 11.

Scholium.

WE will further observe what is of great Use, that of three Lines drawn into or multiplied one by another

ther after what manner soever, a Solid of the same Magnitude is produc'd.

ABC, CAB. BCA.
1. 2. 3.

In the present Scheme the two first Letters design the Base; the third the Altitude. Let us compare the first with the second.

The Base AB is to the Base CA, *per* 1. *l*. 6. as the Side B is to the Side C; that is, reciprocally, as the Height B is to the Height C. Therefore by *p*. 34.

ABC, is equal to CAB.

In the same manner it may be shew'd that the first is equal to the third, and the third to the second.

PROP. XXXVII. Theorem.

Parallelepipeds which are like, and describ'd in the like manner by proportional right Lines, will themselves also be proportional; and conversely.

This is manifest of it self. For the Proportions of the Parallelepipeds, by the 33d of this Book, will be triplicate to those Proportions, (which by the Hypothesis are equal,) which the Lines have betwixt themselves.

The Converse is manifest of it self also.

The Proposition is true of all sorts of like Bodies, which will appear from Book the 12th, to have betwixt themselves a Proportion triplicate to that which the Sides have.

PROP. XXXVIII, XXXIX.

THESE contain nothing remarkable, and are scarce of any Use.

PROP. XL.

THIS is of small Use, and indeed no other than the 28th Proposition in another View.

Scholium.

FRom what hath hitherto been demonstrated, we have the Dimension of Triangular Prisms, and of Quadrangular, or Parallelepipeds; to wit, if the Altitude be multiplied into the Base. As if the Altitude be of 10 Feet, and the Base of a 100 square Feet (now the Base is measur'd by *Schol. p.* 36. or 41. *l.* 1.) multiply 10 by 100, there will arise 1000 cubick Feet for the Solidity of the given Prism.

The Demonstration is easy. For like as a Rectangle ariseth from the Multiplication of one Side by another, so a right Parallelepiped is produc'd from the Height drawn into the Base. Therefore every Parallelepiped is also produc'd from the Altitude multiply'd into the Base; seeing by 31. *l.* 11. it is equal to a right Parallelepiped, constituted upon the same Base with the same Height.

Then seeing the whole Parallelepiped is produc'd from the Height into the whole Base; the half of the Parallelepiped (that is, a Triangular Prism by 28. *l.* 11.) will be produc'd from the Altitude multiplied by half the Base; to wit, the Triangle F E G. See *Fig.* 29.

The

The Elements of EUCLID.

BOOK XII.

With Us the EIGHTH.

WHAT in the foregoing Books we have endeavoured to perform; namely, To bring the Elements of the Mathematicks into a more easy and brief Method, will be to be endeavour'd in this twelfth Book especially; the Doctrine whereof is most necessary, but the Demonstrations are so prolix, that they commonly make Beginners almost to despair. We have so propos'd to our selves to remedy this Evil, that in the mean while we will not depart from the Rigour of Geometrical Demonstration. Which Thing whether or no we have attain'd, the Reader will understand, if he shall compare this of ours with *Euclid*'s Prolixity.

NOW after Euclid *had in the former Book declared the Elements of Solids, and defined the Measures of the most easy Bodies, those, namely, which are terminated with plain Surfaces: In this twelfth Book he considers Bodies bounded with curve Surfaces; to wit, Cylinders, Cones, and Spheres; compares them betwixt themselves; and defines their Measures. This Book is indeed most profitable, because it contains those Principles on which the chief Masters of Geometry, and especially* Archimedes, *have built so many famous Demonstrations, concerning the Cylinder, Cone, and Sphere.*

DEFINITIONS.

Fig. 1. *l.* 12. 1. A Pyramid is a Solid (Z L) comprehended under the Triangles (A L C, C L F, F L B, B L A) plac'd from one Plane (Z) to one Point (L).

The Plane Z is call'd the Base, and may be either a Triangle or Quadrangle, or any other Figure; from each of the Sides whereof there arise Triangles meeting together in the Point L, which is call'd the Vertex or Top.

As the Triangle amongst rectilinear plane Figures, so the Pyramid amongst solid ones is the first and most simple.

Fig. 2, 3. 2. If without the Plane of some Circle (C L) there shall be taken the Point (A), and from it be drawn the infinite right Line (A F) touching the Circle in C; and this Line (the Point (A) remaining fix'd) be turn'd about the Circumference of the Circle, until it returns thither from whence it began to be moved; the Surface describ'd by the right Line (A C F) is term'd a conical Surface, and the Body which is contain'd under this Surface, and the Circle (C L) is call'd a Cone.

The Vertex of the Cone is (A).

The Circle (C L) is the Base of the Cone.

The right Line (A B) drawn from the Vertex to the Centre of the Base is the Axis of the Cone.

The Side of the Cone is the right Line (A C) drawn from the Vertex to the Circumference of the Base, which that it is wholly in the Surface of the Cone, is manifest from the Production of the Figure.

* *Fig.* 2. A right * Cone, is, when the Axis (A B) is perpendicular to the Base.

† *Fig.* 3. A scalene † or oblique Cone, is, when the Axis (A B) is not perpendicular to the Base.

A right Cone is also made by a right-angled Triangle (C B A) turn'd round about one of the perpendicular Sides (A B). See *Fig.* 2.

Fig. 4, 5. 3. If an infinite right Line (C O F) be turn'd about two Circles (C L, O Q) equal and parallel, until it returns to that Place from whence it began to be mov'd, and remains always, whilst it is mov'd, parallel to itself, the Surface describ'd by the right Line (C O F) is call'd

a Cylindrical Surface ; and the Body which is contain'd under this Surface, and the two Circles, is call'd a Cylinder.

The Bases of the Cylinder are the Circles (C L, O Q); The right Line (A B) which connects the Centers of the Bases, is call'd the Axis. The right Line (O C) in the Surface of the Cylinder, touching both the Bases, is called a Side of the Cylinder.

A right Cylinder, is, when the Axis is perpendicular to the Base. *Fig.* 4.

A scalene or oblique Cylinder, is, when the Axis is not perpendicular to the Base. *Fig.* 5.

A right Cylinder is also made by a Rectangle (O C B A) turn'd round about one Side (B A). See Fig. 4.

4. Like Cones and Cylinders are those, which have their Axes (A K, Z O) and the Diameters of their Bases (B F, Q R) proportional. *Fig.* 20, 21.

5. A Sphere is a Solid contain'd under one Surface, unto which Surface all the right Lines that are drawn from a certain Point within the Figure, are equal amongst themselves. That Point is call'd the Centre. The Diameter of the Sphere is a right Line drawn thro' the Centre unto the Surface on both Sides.

A Sphere is produced if a Semicircle be turn'd about its Diameter (A F) which remains in the mean while unmov'd. *Fig.* 6.

6. Magnitudes inscrib'd in or describ'd about some Figure, whether they be greater or lesser than the Figure, are then said to *end* in the Figure, when they will at the last differ from it by a Quantity less than any given one whatsoever, or how small soever.

Therefore if those Magnitudes which are inscrib'd in some Figure, will at last fall short of it by a Deficiency less than any given one whatsoever, the Magnitudes inscrib'd are said to *end* in the Figure ; and if those which are circumscrib'd about some Figure, will at last exceed it by an Excess less than any given one whatsoever, they shall be said to *end* in the Figure.

PROPOSITION I. Theorem.

Fig. 6, 7. *THE Proportion of like Polygons inscrib'd in a Circle, is duplicate to the Proportion of the Diameters (A F, I C).*

Let A O, B F; I R, L C, be drawn. Because the Polygons are suppos'd to be like, the Angles (O B A, R L I) will *(per Defin.* 1. *l.* 6.) be equal; and the Sides O B, B A, proportional to the Sides R L, L I. Therefore in the Triangles O A B, R I L (*per* 6. *l.* 6.) the Angles O and R are equal. Therefore also the Angles B F A and L C I, which stand upon the same Arches B A, L I, are (*p.* 21. *l.* 3.) equal. But the Angles F B A, C L I, in Semicircles, are (*per* 31. *l.* 3.) right ones. Therefore the other Angles (*p. Coroll.* 9. *pr.* 32. *l.* 1.) B A F, L I C, are equal. Therefore because the Triangles F A B, C I L, are equiangular to each other, they are (*p.* 4. *l.* 6.) like; and B A will be to L I, as A F to I C. Now because by the Hypothesis the Polygons are like, their Proportion will be duplicate (*p.* 20. *l.* 6.) to the Proportion of the Sides B A, L I; that is, as I have already shew'd, duplicate to the Proportion of the Diameters, A F, I C. *Q. E. D.*

Corollary.

Fig. 6, 7. THE Circumferences of like Polygons inscribed in a Circle are betwixt themselves as the Diameters.

Seeing it hath already been shew'd, that A B is to L I, as A F is to I C, O B will also be to R L, as A F to I C: And so of the rest of the Sides. Therefore all the Sides together will be to all the Sides together, that is, one Circumference to another, as A F is to I C.

A Lemma.

Fig. 8. POlygons inscrib'd in a Circle end in a Circle. Inscribe a Square, as A C B D, Seeing this is half

(per

(*per Schol. p.* 6, and 7. *l.* 4.) of the Square which is circumscrib'd, it will be greater than half of the Circle. Wherefore if this be taken out of the Circle, there will be taken out of it more than half. Then each Arch being bisected in E, K, I, H, inscribe an Octagon: And let FG touch the Circle in E, which FG let BC, DA meet in G and F; CF will be a Parallelogram, of which seeing the Triangle CEA (*per* 41. *l.* 1.) is half, this will be more than half of the Segment CEA. In the same manner each of the Triangles AKD, DIB, *&c.* is more than half each of the Segments. Therefore all the Triangles are more than half all the Segments. Therefore if you take these out of those, that is, out of the Remainder of the Circle, more than half will be taken away. In the same way of arguing, if there be inscrib'd in the Circle, Polygons of Sides always double in Number; I can shew that there will always be taken out of the Remainder of the Circle more than half. Therefore the Remainder must at last be less than any given one whatsoever; and consequently the inscrib'd Polygons will at last fall short of a Circle by a Quantity less than any given one whatsoever; that is, (*per Defin.* 6. *l.* 12.) will end in a Circle.

PROP. II. Theorem.

THE Proportion of Circles is duplicate to the Proportion of their Diameters. Fig. 6, 7.

The Proportion of Polygons inscrib'd in a Circle without End is (*per* 1. *l.* 12.) duplicate to the Proportion of the Diameters. But Polygons (by the foregoing *Lemma*) inscrib'd in a Circle infinitely, at last end in the Circle. Therefore the Proportion of Circles is also duplicate to the Proportion of the Diameters.

PROP. III, IV.

ARE Prolix, and hard for young Beginners, and have no other Use, than that they serve to the Demonstration

monstration of the Fifth, which we shall demonstrate much more easily without them.

Lemmata, *or preparatory Propositions to* Prop. V.

Lemma I.

Fig. 9. If two triangular Pyramids be cut with Planes (O S E, R X Z) parallel to the Bases (A B C, I Q V), which same Planes divide the Sides (C F, Q L) proportionally in (E and Z:) then O S E, R X Z will be betwixt themselves as the Bases (A C B, I Q V).

Because the parallel Planes O S E, A B C, are cut by the Planes B F C, A F B, A F C, the common Sections S E, B C, and O S, A B, and O E, A C, will be (*per* 16. *l.* 11.) parallel. Wherefore the Angles O S E, A B C, and S O E, B A C, and O E S, A C B, two and two, are (*per* 10. *l.* 11.) equal. Wherefore the Sections O S E, A B C, are like (*per* 4. *l.* 6.) In the same manner I might shew that the Sections R X Z, I V Q, are like. Therefore (*per* 19. *l.* 6.) the Proportion of the Section A B C, to the Section O S E is duplicate to the Proportion of the Side B C, to the Side S E; and the Proportion of the Section I V Q to R X Z is duplicate to the Proportion of V Q to X Z. But the Proportions of B C to S E, and of V Q to X Z are the same; (for B C is to S E (by *Coroll.* 1. *per* 4. *l.* 6.) as C F to E F; that is, by the Hypothesis, as Q L to Z L; that is, (by the same *Coroll.*) as V Q to X Z). Therefore the Proportion of A B C to O S E is the same with the Proportion of I V Q to R X Z. *Q. E. P.*

Lemma II.

Fig. 10. Prisms inscrib'd infinitely in a Pyramid (Z C A F) which hath a triangular Base, end in the same Pyramid.

Let the Side of the Pyramid be divided into a certain Number of equal Parts A B, B G, G F, and thro' B and G there being made the Sections G D N and B E P parallel to the Base Z A C; let the triangular Prisms B E P M A O and G D N K B Q be understood to be inscrib'd

in the Pyramid. These then being continued without the Pyramid, let there be understood to be describ'd about the Pyramid the Prisms CIBA, PXGB, NHFG. The Excesses of the circumscrib'd Prisms above the inscribed ones are the Solids IM, XK, HG, which taken together are equal to the Prism CIBA: For HG (*per* 25. *l.* 11.) is equal to DB; and consequently HG with XK are equal to PXGB, that is, (by the same) to MEBA. Therefore the three HG, XK, IM, are equal to the whole CIBA. But if AF be divided without End into more equal Parts, and consequently the Number of Prisms be infinitely increas'd, AB will become less than any given Line. Therefore (as it is manifest from *p.* 25. *l.* 11.) the Prism CIBA will become less than any given one. Therefore the Excess of the circumscrib'd Prisms, (and much more of the Pyramid ZCAF which is part of the Prisms circumscrib'd about it) above the inscribed Prisms will be less than any given Prism. Therefore the inscrib'd Prisms (by *Defin.* 6. *l.* 12.) end at last in a Pyramid. *Q. E. D.*

PROP. V. Theorem.

TRiangular Pyramids of the same Height have that Proportion betwixt themselves, which their Bases (AQR, ESX) have. Fig. 11.

Let the equal Altitudes of the Pyramids be represented by the Side AP, EZ; which on both Sides let be divided into as many equal Parts as you will, but so that they be of the same Number; and let there be made thro' the Points of the Divisions, Sections parallel to the Bases: let triangular Prisms, of the same Number and the same Height, be understood to be inscrib'd in both Pyramids. And now because the Prisms, LA, IE, are of the same Height, the Prism LA will be to the Prism IE (by *Coroll.* 1. *p.* 34. *l.* 11.) as the Base LOB is to the Base INK; that is, (by *Lemma* 1.) as the Base QRA is to the Base SXE. In the same manner I might shew that each of the Prisms inscrib'd in the Pyramid QPAR, is to each inscrib'd into the Pyramid SZEX, as the Base QAR is to the Base SEX. There-

Therefore all of them together are to all of them together, as Base is to Base. Wherefore seeing they at last end (*per Lem.* 2.) in the Pyramids themselves, the Pyramids themselves also will be as their Bases. *Q. E. D.*

PROP. VI. Theorem.

Fig. 12, 13. *ALL Pyramids whatsoever, which are of equal Height have that Proportion betwixt themselves which their Bases (A B, C F O), have.*

Let their Bases be resolv'd into Triangles A, B, C, F, O; and the whole Pyramids into triangular Pyramids. The Pyramid A X is to the Pyramid O Z (by the foregoing) as A is to O; and the Pyramid B X is to the Pyramid O Z, as B is to O (by the same). Therefore the Pyramids A X, B X, together (that is, the whole Pyramid A B X) are to the Pyramid O Z, as A, B together are to O. By the same Argumentation the Pyramid A B X is to the Pyramid F Z (by the foregoing), as A, B are to F: And A B X is to C Z, as A, B is to C. Therefore A B X is to the three O Z, F Z, C Z together; that is, to the whole Pyramid O F C Z, as A, B, together is to O, F, C together. *Q. E. D.*

PROP. VII. Theorem.

Fig. 14. *EVery Pyramid is the third Part of a Prism which hath the same Base and Height.*

First, let the triangular Pyramid B G A C have the same Base and Height with the Prism B A C F E O: Let B F, A O, A F, be drawn. The Triangles B F C, B F O are (*per* 34. *l.* 1.) equal. Therefore the Pyramid B F C A, is equal to the Pyramid B O F A. For the same Reason O E A F, is equal to the Pyramid O B A F; that is, to the Pyramid B O F A, for they are the same Pyramids. Therefore B F C A, and O E A F, are also equal. Therefore all three B F C A, O E A F, O B A F, or B O F A, are equal. Therefore the three together

gether are triple of one B F C A. But those three constitute the Prism B A C F E O. That Prism therefore is triple to the Pyramid B F C A; that is, (*per* 5. *l.* 11.) to B G A C. *Q. E. D.*

Then let any Pyramid whatsoever have the same Base and Height with the Prism A E F H: the Lines B C, B O, B E, and N I, N G, N H, being drawn, resolve the Prisms into triangular Prisms, and the Pyramid into triangular Pyramids. Which being done, the Demonstration is manifest from the first Part: For each Part of the Prisms will be triple of each Part of the Pyramids. And consequently the whole Prism will be triple to the whole Pyramid. *Q. E. D.* *Fig.* 15.

PROP. VIII. Theorem.

THE Proportion of like Pyramids (O A C B, K H I N) is triplicate to that which the homologous Sides (A B, H N) have to each other. *Fig.* 16.

First let them be Triangular: The Parallelograms A M and H Q being perfected, set upon them the Parallelepipeds A G, H L, in the Height of the Pyramids; which, seeing the Pyramids are like, will also (as appears from *Defin.* 9. *l.* 11.) be like. Then let E F, R P, be drawn; and thro' E F, C B, as likewise thro' R P, I N, the Parallelepiped will be cut (*per* 28. *l.* 11.) into two equal Prisms; each of which will be triple to the Pyramids O A C B, and K H I N (by the foregoing). Therefore both together, that is, the whole Parallelepipeds A G, H L will be Sixfold of the Pyramids. Therefore the Pyramids are proportional to the Parallelepipeds. But (*per* 33. *l.* 11.) the Proportion of these each to other is triplicate to the Proportion of the Sides A B, H N. Therefore so likewise is the Proportion of the Pyramids.

But if the like Pyramids shall be polygonal, let them be resolv'd into the triangular ones A R, B R, C R, and O K, E K, F K. You may from 20. and 5. *l.* 6. and *Defin.* 9. *l.* 11. easily shew that A R is like to O K, and B R to E K, and C R to F K. Therefore by the former Part, the Proportion of the Pyramids A R, O K, is triplicate to the Proportion of I M to P Z: And the Proportion *Fig.* 17.

portion of the Pyramids BR and EK is triplicate to the Proportion of MX to SZ; that is, again by the Hypothesis, of IM to PZ; and the Proportion of the Pyramids CR, FK is triplicate to the Proportion of XC to ST; that is, again of IM to PZ. Seeing therefore the Proportion of each to each is triplicate to the Proportion of IM to PZ, the Proportion also of all to all (that is, the Proportion of the whole Pyramid ABCR to the whole OEFK) will be triplicate to the Proportion of IM to PZ. *Q. E. D.*

PROP. IX. Theorem.

Fig. 18, 19. *Equal Pyramids have their Bases and Altitudes reciprocally proportional; and those which have them so, are equal.*

Part I. First let the Pyramids be triangular BACO, KHNL: The Parallelograms BE, HR, being perfected, upon these set the Parallelepipeds, BF, HP. These will be (as was shew'd in the foregoing) sixfold of Pyramids which are by the Hypothesis equal, and consequently will be equal betwixt themselves. But now the Altitudes of these Parallelepipeds HK, BA, are the same with those of the Pyramids; and the Bases BE, HR, are double to the pyramidal Bases (*per* 34. *l.* 11.) BCO, HNL, and consequently proportional to them. Seeing therefore by Reason of the Equality of the Parallelepipeds, as BE is to HR, so (by the same) is reciprocally HK, to BA; it will also be that as the Base BCO is to the Base HNL, so reciprocally is the Altitude HK to the Altitude BA. *Q. E. D.*

But if the Pyramids have polygonal Bases, let them be reduced into triangular ones, retaining the same Altitudes; and these will be equal to those by the 6th. But the Pyramids thus reduc'd, have, as we have now demonstrated, their Bases and Altitudes reciprocally proportional. Therefore the given polygonal Pyramids also have their Bases and Altitudes reciprocally proportional. *Q. E. D.*

Part II. Because it is now suppos'd, that B C O is to H L N, as H K is to B A; B E will also be to H R, as H K is to B A. Therefore the Parallelepipeds B F, H P, are (*per* 34. *l.* 11.) also equal. Therefore their sixth Parts also, to wit, the Pyramids B A C O, H K N L, are equal. *Q. E. D.*

Corollaries.

WHAT has been demonstrated of Pyramids in *pr* 6, 8, 9. does also agree to all Prisms whatsoever; seeing these are (*per* 7. *l.* 12.) triple to Pyramids which have the same Bases and Altitudes. Therefore,

1. In Prisms of the same Height, their Proportion is the same as that of their Bases. For this was shew'd of Pyramids, *pr.* 6.

2. The Proportion of like Prisms is triplicate to the Proportion of their homologous Sides. For this was shew'd concerning Pyramids, *pr.* 8.

3. Equal Prisms have their Bases and Altitudes reciprocally proportional; and those which have them so are equal. For this is shew'd of Pyramids, *pr.* 9.

It is strange that these Things were pass'd over by *Euclid*, seeing they are the chief Things which can be deliver'd concerning rectilinear Solids.

Scholium.

FROM what has been hitherto demonstrated is deduc'd the Method of measuring any Prisms or Pyramids whatsoever.

The Solidity of a Prism is produc'd from the Altitude multiplied into the Base; and that of a Pyramid from the third Part of the Altitude multiplied by the Base.

As if the Altitude of a Prism be of 5 Feet, but the Base contains 25 square Feet; multiply 25 by 5, and there arises 125 cubick Feet for the Solidity of the Prism.

For let there be a polygonal Prism as A H. And let the Triangle B A C be understood to be equal to its Base A E, and upon B A C the Prism B E to be set at equal Height *Fig.* 15, 14.

Height with A H. The Prisms B E, A H, will be (by *Coroll.* 1. foregoing) equal. But the Prism B E (by *Schol. p.* 40. *l.* 11.) is produc'd from its Altitude drawn into the Base B A C; that is, into A E, by Construction. Therefore the Prism A H also is made of its Base A E multiplied by its Height, which is supposed to be equal to the Height of the Prism B E.

From hence and from 7. the Demonstration of the second Part is also manifest.

A Lemma to Prop. 10.

PYramids and Prisms which are inscrib'd in Cones and Cylinders infinitely, do at last end in the Cones and Cylinders.

This is demonstrated as the *Lemma* of *Prop.* 2. with the help of *Prop.* 6. and of *Coroll.* 2. after *Prop.* 9. if as there Planes inscrib'd in a Circle, so here Prisms and Pyramids which stand upon those Planes as their Bases, be continually taken away from the Cones and Cylinders.

PROP. X. Theorem.

Fig. 20. *EVery Cone is a third Part of a Cylinder having the same Base and Height.*

Let a regular Polygon of as many Sides as you please be understood to be inscrib'd in the Base C L, and upon it as the Base, for a Cone let a Pyramid, and for a Cylinder a Prism be inscrib'd. The Pyramid (*per* 7. *l.* 12.) will be a third Part of the Prism. And if again in the Circle a Polygon of twice as many Sides be inscrib'd, and upon it be inscrib'd for a Cone a Pyramid, but for the Cylinder a Prism; the Pyramid will again be a third Part of the Prism. And thus it will always be. Wherefore seeing Pyramids end in a Cone, and Prisms in a Cylinder, the Cone also will be a third Part of the Cylinder. *Q. E. D.*

PROP.

PROP. XI. Theorem.

Cones of equal Height (B A F, Q X R) are as their Bases (C L, S E). The same Thing belongs to Cylinders of equal Height also. Fig. 20, 21.

Pyramids inscrib'd into Cones of equal Height, are as their Bases (*per* 6. *l.* 12.) But Pyramids do at length end in Cones. Therefore Cones also are as their Bases. And seeing Cylinders are threefold of Cones, which have the same Base and Altitude with them, they also will be as their Bases. *Q. E. D.*

Coroll.

IN the same manner it may be demonstrated, that Prisms and Cylinders also of equal Height are betwixt themselves as their Bases; yea, that all cylindrical Bodies of the same Altitude; that is, which are produc'd from whatsoever Planes multiplied by the same Altitude, are betwixt themselves as their Bases. You may reason in the same manner of Pyramids and Cones of equal Altitude, and of all conical Bodies whatsoever.

PROP. XII. Theorem.

THE Proportion of like Cones (B A F and Q Z R) is triplicate to the Proportion of the Diameters (B F and Q R) which are in the Bases. The same Thing is to be said of like Cylinders. Fig. 20, 21.

In the Bases of the like Cones let regular Polygons be inscrib'd, which Polygons consequently will be like. The Pyramids which are inscrib'd upon these Polygons will also be like; as may be easily shew'd. Therefore their Proportion is triplicate (*per* 8. *l.* 12.) to the Proportion of the Sides B L, Q E; that is, to the Proportion of the Diameters B F, Q R. Wherefore seeing the Pyramids end in Cones, the Proportion also of the Cones

is triplicate to the Proportion of the Diameters B F, Q R. *Q. E. D.*

The Theorem is manifest of Cylinders, seeing they are triple to Cones.

PROP. XIII. Theorem.

Fig. 22. *IF a Cylinder (B I) be cut with a Plane (R L) parallel to the Bases (B Q, C I); one Part (B L) shall be to the other Part (R I), as one Segment of the Axis (A O) is to the other Segment of the Axis (O F.)*

This Proposition is demonstrated as the first of *l.* 6.

The Theorem is in the same manner true of the Superficies.

PROP. XIV. Theorem.

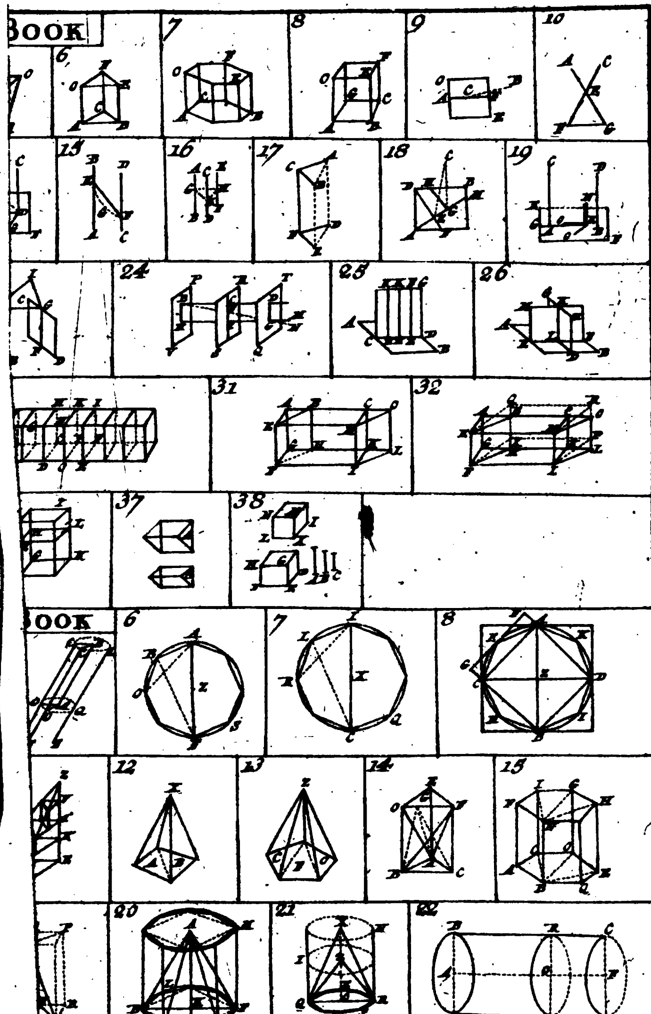
Fig. 23, 24. *CYlinders (A R and C I) of equal Bases (M Q, G B) are as their Altitudes (L Z, S F). The same Thing happens to Cones.*

Cut off from the higher Cylinder A R the Cylinder A O, whose Height L E is the same with S F. Therefore (*per* 11. *l.* 12.) the Cylinders A O, C I, are equal. Seeing therefore the Cylinder A O, is to the Cylinder A R, (by the foregoing) as L E is to L Z; C I also shall be to A R as L E is to L Z; that is, (because L E and S F are equal, by Construction) as S F to L Z. *Q. E. D.*

Corollary.

THE Theorem is also true of Prisms, and likewise of Pyramids, and the Demonstration altogether alike. But of Prisms the thing is demonstrated from *Corol.* 1. *p.* 9. *l.* 12. and 25. *l.* 11. and its *Corol.* Of Pyramids from this, and from *p.* 7. *l.* 12.

PROP.



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N 2 PROP.

PROP. XV. Theorem.

EQual Cylinders (*A R, D F*) *have their Bases and Altitudes reciprocally proportional; and if they have them so they are equal. The same Thing is true of Cones.* Fig. 24, 25.

This is demonstrated as *Prop.* 34. *l.* 11. only for 32, and 25. *l.* 11. there cited, there must be cited here *Prop.* 11, and 13. *l.* 12.

Scholium.

WHereas *Euclid* hath said nothing of compound Proportion in Bodies, we shall briefly demonstrate it in this Place.

1. A Cylinder hath to a Cylinder, and a Prism to a Prism, a Proportion compounded of the Proportions of the Bases and Altitudes.

Let F D and A R be Cylinders of different Altitudes (for in those of equal Altitude the Thing is manifest.) From the higher cut off A O of equal Height with F D. And let the Proportion be thus; as the Base V T is to the Base M Q, so F N to X; and as the Altitude N D or B O is to the Altitude B R, so is X to Z. We must therefore shew, that the Cylinder F D is to the Cylinder A R, as F N is to Z. The Cylinder F D is to the Cylinder A O (*per* 11. *l.* 12.) as the Base V T is to the Base M Q; that is, (by Construction) as F N is to X; but the Cylinder A O is to the Cylinder A R (*per* 13. *l.* 12.) as B O to B R; that is, (by Construction) as X to Z. Therefore by Proportion of Equality the Cylinder F D is to the Cylinder A R, as F N to Z. Fig. 25, 24.

The Proposition may be demonstrated in the same manner of Prisms, but from *Coroll.* 1. *pr.* 9. and *Coroll. pr.* 14.

2. A Cone also hath to a Cone, and a Pyramid to a Pyramid, a Proportion which is compounded of the Proportions of Base to Base, and Altitude to Altitude.

For (by *Prop.* 10, and 7. *l.* 12.) they are third Parts of Cylinders and Prisms.

PROP. XVI, XVII.

THese *Propositions, the most prolix of all other, have no other Use than to serve to the demonstrating* Prop. 18. *which we shall demonstrate in another more easy Way.*

Lemma to Prop. 18.

Fig. 16. CYlinders inscrib'd in an Hemisphere end in the Hemisphere. Let P Z Y be the greatest Semicircle of the Hemisphere; and let the Radius A Z be perpendicular to the Diameter P Y. Cut A Z into a certain Number of equal Parts, A M, M N, N Z; and there being drawn thro' the Points of the Divisions M, N, the perpendicular Lines B O, &c. Let there be inscrib'd in the Semicircle, the Rectangles O B R K, E D H S; which afterwards being continued without the Semicircle, let there be understood to be describ'd about the Semicircle, the Rectangles F T Y P, L V B O, Q X D E: They will all of them be of the same Height, and the Excesses of the circumscribed ones above those which are inscribed will be the Planes F K, L S, X E, V H, T R, which taken together make the Rectangle F T Y P. For because X E is equal to D S, those L S, V H, X E together, will be equal to the Rectangle L B, that is, O R. Wherefore if you add on both Sides the Planes F K, T R, all those F K, L S, X E, V H, T R, taken together, will be equal to the Rectangle F T Y P. If now the Semicirele with the Rectangles be understood to be turn'd about the Radius A Z, which is in the mean while unmov'd, the inscribed Rectangles E H, O R, will produce Cylinders inscrib'd in the Hemisphere; and the circumscrib'd Rectangles will produce Cylinders circumscrib'd about the Hemisphere, standing one upon another; and as the Excesses of the circumscribed Rectangles above the inscribed ones, was the Rectangle F Y; so likewise the Excesses of the circumscribed Cylinders above the inscrib'd ones, will be the Cylinder which is produced from the Rectangle F Y. But now the Altitude

tude of this Cylinder will be made less than any given Height; and consequently (as is manifest from 13. *l.* 12.) it self will grow to be less than any given Cylinder, if, the Radius being divided into more equal Parts without end, the Number of Rectangles, and from thence of Cylinders, be infinitely increas'd. Therefore the Excess of the circumscrib'd Cylinders, and much more of the Hemisphere it self, which is only a Part of the circumscrib'd Cylinders above the inscrib'd ones, will at last become less than any given one. Therefore (by *Defin.* 6. *l.* 12.) Cylinders infinitely inscrib'd in an Hemisphere, do at length end in the Hemisphere it self. *Q. E. D.*

Corollary.

IN the same manner it will be demonstrated, that Cylinders inscrib'd in a Cone, Conoid, Spheroid, *&c.* do at last end in the same.

PROP. XVIII. Theorem.

THE Proportion of Spheres is triplicate to the Proportion of their Diameters (*B K, R Z*). *Fig.* 27.

The Radius's A B, Y R, being divided into as many equal Parts as you will, but of an equal Number, and there being drawn thro' the Points of the Divisions Perpendiculars, *&c.* let Rectangles of an equal Number be understood to be inscrib'd in the greater Semicircles of the Spheres, which Rectangles being turned about, the unmov'd Radius's A B, Y R, will be conceiv'd to inscribe in both the Hemispheres a like Number of Cylinders standing one upon another. Now because K C is (*per Coroll. p.* 13. *l.* 6.) to C F, as C F is to C B; the Proportion of K C to C B (by *Defin.* 10. *l.* 5.) will be duplicate to that of K C to C F, that is, to the Proportion of F C to C B. In like manner the Proportion of Z E to E R will be duplicate to the Proportion of X E to E R. But by the Construction K C is to C B, as Z E is to E R. Therefore F C also is to B C, as X E to E R. But B C by the Construction is to C O, as R E to E S. Therefore by Equality F C is to C O, as X E is

is to E S. Therefore (by *Defin.* 4. *l.* 12.) the Cylinders F L, X Q, are like, and consequently their Proportion is (*per* 12. *l.* 12.) triplicate to the Proportion of their Diameters, F I, X V, or of the Semidiameters F C, X E, which are in the Bases. But the Proportion of F C to X E is the same with the Proportion which is betwixt the Diameters of the Spheres B K, R Z; (for as I have already shew'd, F C is to X E, as C O is to E S; that is, as B K is to R Z, which by the Construction are Equi-multiples of those C O, E S.) Therefore the Proportion of the Cylinders F L, X Q is triplicate to the Proportion of the Diameters B K, R Z. In the same manner we might demonstrate that each Cylinder inscribed in one Hemisphere, bears to each Cylinder inscribed in the other Hemisphere, a Proportion triplicate to the Proportion of the Diameters B K, R Z. Therefore also the Proportion of all together to all together (by 12. *l.* 5.) is triplicate to the Proportion of the Diameters B K, R Z. Wherefore seeing the Aggregates of the Cylinders do at length end in their Hemispheres, the Proportion of the Hemispheres also, and consequently of the Spheres will be triplicate to the Proportion of their Diameters. *Q. E. D.*

Corollary.

THerefore the Proportion of the Diameters being known, the Proportion of the Spheres becomes known likewise. As if the Diameter of the lesser be one Foot, that of the greater ten Feet; let the Proportion of one to ten be continued thro' four Terms, 1, 10, 100, 1000; as 1 the first is to 1000, the 4th Term, so is the lesser Sphere to the greater.

The Dimension of Cones, Cylinders, and of the Sphere, will be exhibited in the following Book out of *Archimedes.*

Scholium.

AS like plain Figures are increas'd or diminished in any given Proportion by one mean Proportional, so like Bodies are increas'd or diminish'd by two mean Proportionals.

Let

Let a Sphere or Cube, or any other Body whatſoever, be given, whoſe Radius or Side is A. Likewiſe let any Proportion whatſoever of A to B be given, as the double, or 2 to 1. A Body is to be diſcover'd both double to the given one and like to it.

Betwixt the Terms of the given Proportion A and B, let there be found two mean Proportionals X, Z, according to what was taught in the *Scholium* of *Prop.* 13. *l.* 6. A Sphere whoſe Radius is X, or other Body like to the given one which is made upon the Side X, will be double to the given one.

For like Bodies whoſe Radius's or Sides are A and X, have betwixt themſelves the Proportion which is triplicate to the Proportion of A to X, (by *Coroll. Prop.* 9, and by *Prop.* 12. and 18. *l.* 12.) that is, the ſame (*per. Defin.* 10. *l.* 5.) which A hath to B.

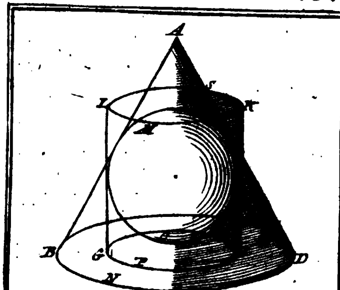
And this is that moſt celebrated Problem which from *Apollo* and *Delos* is called the *Deliacal* Problem; becauſe at the time of a moſt grievous Peſtilence, which waſted *Athens*, being conſulted, he gave Anſwer, that the Peſtilence would ceaſe, if his Altar, which was of a cubical Form, were doubled. Thus *Valerius Maximus* l. 8.

THEOREMS
Selected out of
ARCHIMEDES:
By ANDREW TACQUET,
OF THE
SOCIETY of JESUS;

And Demonstrated in a more Easy and Compendious Way.

To which are added,

Some other Propositions, newly invented, by the same *Andrew Tacquet.*



Una tribus Ratio est

LONDON,
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To the Reader.

ALBEIT there have appeared very many most excellent and admirable Men in the Mathematical Sciences; yet the chief Glory of all hath always by a certain common Consent, been given to Archimedes *of* Syracuse. *Tho indeed More there are who commend than who read him; more who admire than understand him. The Causes of which Neglect seem to be these, the Bulk and Scarceness of Copies, some Obscurity of the Translation, which is directly made out of the* Greek *Language, together with the Prolixity and Difficulty of his Demonstrations. I judged therefore that it would be for the Profit of studious Learners, if after my Illustration of the Elements, I should subjoin these* Theorems *which had been selected by me out of* Archimedes, *and demonstrated in a much easier and briefer Way. Furthermore, I have selected those, which bring along with them both more of Admiration and of Benefit; and have in my Demonstration took such a Method, that, I hope, he who understands the Elements will, without any great Labour, comprehend these most excellent Inventions of the Prince of Geometricians. I have also added at the End, thirteen Propositions, and thereby enlarged the Doctrine of* Archimedes, *concerning the Sphere and Cylinder: Where amongst other Things, I demonstrate, that the sesquialteral Proportion is continued in the Three Bodies, a Sphere, Cylinder, and equilateral Cone; both the latter being inscrib'd about the Sphere. Moreover I have added divers Things here and there, amongst which the 12th* Proposition, *and the* Corollaries *of* Prop. 14. *are the chief; and several* Scholiums.

liums. *Make use of these Discoveries whosoever thou be'st, that art a Candidate of* Geometry; *and how much thou hast improv'd in* Euclid, *make Proof of in* Archimedes. *And when thou perceivest thy self to be fix'd and rais'd upwards in the Contemplation of the most noble Truths, raise up thy Mind, while it is thus already lifted up from these lower Things, yet higher, and direct it to that Truth which is Original, Eternal, Immense, and is no other than GOD; by the ineffable Vision of whom, I trust we shall hereafter be made eternally Happy. Farewel.*

THEOREMS
Selected out of *ARCHIMEDES*.

DEFINITIONS;
Or an Explanation of certain Terms.

LET there be a Circle BECG, whose Centre is A, its Diameter BC, which let the right Line EG cut at right Angles, (but not thro' the Centre) in D. Let there be drawn from the Centre the Radius's AE, AG. This being suppos'd. *Fig.* 23. Of the Table out of *Archimedes*.

NOTE, 1. That a Sector of a Sphere is that which is produced from the Sector of the Circle AECG, or AEBG, turn'd round about the Diameter BC.

2. That a Segment or Portion of a Sphere is that Part of it which is produc'd from the Segment of the Circle ECG or EBG turn'd round about the same Diameter BC.

3. The Vertex or Top of the Spherical Portion EBG is the Extremity B of the unmov'd Diameter; the Basis, the Circle describ'd by EG; the Axis, that Part of the Diameter BD, which is intercepted betwixt the Top B, and D the Centre of the Base.

4. When I name the Superficies of a Spherical Portion, or of a Body inscrib'd in it, or of a Cone, I always understand it without the Base; and when I say the Superficies of a Cylinder, I mean likewise without the Bases; unless the Word [whole] be adjoin'd to [Superficies]; for then the Bases also are to be taken in.

Again,

Again, when I treat of Cylinders or Cones, I speak of no other than right ones.

Axioms.

Fig. 1, 16. 1. THE Circuit of a Polygon inscrib'd in a Circle is less than the Circumference of the Circle.

Fig. 1. 2. The Circuit of a Polygon describ'd about a Circle is greater than the Circumference of the Circle.

Fig. 16. 3. And if a Polygon inscrib'd in a Circle, be turn'd about the Diameter (A E) together with the Circle, the Superficies of the Body produc'd by the Polygon, will be less than the Superficies of the Sphere. And if a Polygon circumscrib'd about a Circle, be turn'd about the Diameter, together with the Circle, the Superficies of the Body produc'd by the Polygon will be greater than the Superficies of the Sphere.

Fig. 17. 4. In like manner the Circuit of a Polygon inscrib'd in a Segment of a Circle (D A F) is less than the Circumference of the Segment. And if a Polygon inscrib'd in the Segment, be together with the Segment (A O) turned round; the Superficies of the Body produc'd by the Polygon will be less than the Superficies of the Spherical Portion (D A F).

Fig. 3, 6. 5. The Superficies of a Prism inscrib'd in a Cylinder is less than the Superficies of the Cylinder; but the Superficies of the Prism which is circumscrib'd is greater.

Fig. 4, 8. 6. And the Superficies of a Pyramid inscrib'd in a Cone, is less than the Superficies of the Cone; but the Superficies of a circumscrib'd Pyramid is greater.

PROPOSITIONS I, II.

A*RE not necessary.*

PROP. III. Theorem.

T*HE Circuits of Polygons circumscrib'd about and inscrib'd in a Circle, do at last end in the Circumference of the Circle. In like manner the Polygons themselves do at last end in the Circle.*

If,

If, to wit, the Arches being bisected without end, more and more Sides be circumscrib'd about and inscrib'd in the Circle. Fig. 1. Tabl. Archimed.

Part I. Let there be understood to be inscrib'd in and describ'd about a Circle, regular Polygons; whether it be done so as is set down, *Prop.* 12. *l.* 4. or as in the present Figure, the Thing will be the same. It is manifest (*per Coroll.* 1. *p.* 4. *l.* 6.) that FI is to CE (that is, the whole Circuit circumscrib'd, unto the whole Circuit inscrib'd) as IA is to CA. But IC the Excess of the right Lines IA above CA, becomes at length less than any given Line, if more and more Sides be understood to be infinitely circumscrib'd and inscrib'd; therefore also the Excess of the Circuit circumscrib'd above that which is inscrib'd, will at length become less than any given Line. Therefore much more the Excess of the Circuit circumscrib'd above the Circumference of the Circle will be less than any given one. In like manner, because I have already shew'd the Defect of the Circuit inscrib'd, whereby it falls short of that which is circumscrib'd, to be less than any given Line: Therefore much more will the Defect of the Circuit inscribed, whereby it falls short of the Circumference of that Circle, become less than any given Line. The Circuits therefore, as well that which is inscrib'd, as that which is circumscrib'd, do at length (*Defin.* 6. *l.* 12.) end in the Circumference. Which was the first Part. To demonstrate these Things further is not worth the while, seeing they are manifest enough.

Part II. Because it hath already been shew'd that the Excess of FI above the Side EC becomes at length less than any given Line (for FI is to EC, as IA to CA); therefore also the Excess of the Square of FI above the Square of EC will become at length less than any given Line. But as the Square of FI is to the Square of EC, so (*per* 20. *l.* 6.) is the Polygon circumscrib'd, to that which is inscrib'd. Therefore the Excess of the Polygon circumscrib'd above that which is inscrib'd, will also become at length less than any given one. Therefore much more will the Excess of the Polygon circumscrib'd above the Circle, become at last less than any given one; and consequently, the Defect also of the Polygon inscrib'd, whereby it falls short of the Circle, will at length

length become less than any given Defect. Therefore Polygons as well inscrib'd as circumscrib'd, do at last (*Defin.* 6. *l.* 12.) end in the Circle. Which was the second Part.

PROP. IV. Theorem.

(a) *Per def.* 3. *l.* 4. *Fig.* 1.

A *Regular* (a) *Polygon* (*FINTR*) *circumscrib'd about a Circle, is equal to a Triangle whose Base is the Circuit of the Polygon, and its Height the Radius of the Circle.*

And a regular Polygon inscrib'd in a Circle, is equal to a Triangle, which hath for its Base the Circuit of the Polygon, and for its Height the Perpendicular (*AO*) *let down upon one Side from the Centre.*

Part I. The Radius AB drawn to the Point of Contact is (*per.* 18. *l.* 3.) perpendicular to the Tangent IF. Wherefore if the right Lines AF, AI, AN, &c. being drawn, the Polygon be resolv'd into Triangles; the Radius AB will be the common Altitude of all; and consequently it is manifest that the Triangles are equal. Therefore a Triangle which hath its Base equal to the Sum of the Sides FI, IN, NT, &c. and AB for its Altitude, will (as is manifest from 1. *l.* 6.) be equal to them all, that is, to the whole Polygon circumscrib'd.

Part II. This may be concluded by the same reasoning as the other. [*See Prop.* 14. *Cor.* 3.]

PROP. V. Theorem.

Fig. 2.

A *Circle is equal to a Triangle, which hath for its Base the Circumference, and for its Height the Semidiameter of the Circle.*

Regular Polygons circumscrib'd about a Circle, and Triangles which have for their Bases the Circuit of the Polygon, and for their Altitude the Radius of the Circle, are always (by the foregoing *Prop.*) equal. But Polygons circumscrib'd infinitely about the Circle, end in the

the Circle, (by the 3d of this Book); and in like manner Triangles (as I will shew by and by) which have for their Base the Circuit of the circumscrib'd Polygon, and for their Altitude the Radius A B, at last end in a Triangle which hath the Circumference for its Base, and for its Altitude the Radius A B. Therefore a Circle and a Triangle which hath the Circumference for its Base, and the Radius for its Altitude are equal.

But that Triangles contain'd under the Circuit of the Polygon, and the Radius of the Circle, end at last in a Triangle, which is contain'd under the Circumference and the Radius, I thus shew. Triangles under the Circuit of the circumscribed Polygon and the Radius A B, are to the Triangle which is under the Circumference and the Radius A B (by 1. *l.* 6.) as Base to Base, that is, as the Circuit of the Polygon to the Circumference; since this Triangle and the other have a common Altitude. But the Circuit of the Polygon (by the 3d) ends in the Circumference. Therefore the other Triangles end in this.

Corollaries.

1. FRom this and 41 *l.* 1. it is manifest that a Rectangle under the Radius and half the Circumference is equal to the Circle; that one under the Radius and the whole Circumference is double; that one under the whole Circumference and whole Diameter is quadruple thereto.

2. A Circle is to an inscribed Square, as half the Circumference (C D E) is to the Diameter; but to a Square circumscribed, as the fourth Part of the Circumference is to the Diameter. *Fig.* 5. *l.* 4.

For the Rectangle under C D E, and the Radius C A or C F, that is (by the foregoing *Corollary*) the whole Circle, is to the Rectangle G F C E, to wit, the Rectangle under F G and C F (that is, to the inscribed Square B C D E) as (*per* 1. *l.* 6.) C D E, half the Circumference, is to F G or C E, the Diameter; which was the first Thing. And consequently the Circle is to the double of the Rectangle G F C E, (that is, to F H the circumscribed Square) as C D E is to the double of the

set, more and more without Limit, and so come nearer and nearer for ever to the true Proportion. This hath been perform'd by *Ludolph Ceulen*, *Grimberger*, *Metius*, *Snellius*, and others. The chief Proportions hitherto found I shall here subjoin.

[*Now, since a Tangent of 30 Degrees multiplied by 12, gives the Circuit of a circumscribed Hexagon; and a Sine of 30 Degrees multiplied by 12, gives the Circuit of an Hexagon, which is inscribed: Forasmuch also as in like manner the Tangent of half a Degree multiply'd by 720, yields the Circuit of a circumscrib'd Polygon of 360 Sides; and the Sine of half a Degree, the Circuit of an inscribed Polygon of 360 Sides; and so on for ever: It will not be difficult to understand, by what Means many such Numbers may be found, out of the now given Tables of Sines and Tangents.*]

The first Proportion, which is that of *Archimedes*, is thus:

The Diameter 7

The Circumf. is 22; which is greater than the true.

The Diameter 71

The Circumf. is 223; less than the true one.

The Proportions of 22 to 7, and 223 to 71, if they be reduced to a common Consequent, (which is done after the same manner, in which Fractions are reduced to the same Denomination) will be thus, 1562 to 497, and 1561 to 497.

Therefore the Diameter being suppos'd 497 Parts, the Circumference greater than the true one will be 1562; and the Circumference less than the true 1561.

Both of them therefore differ from the true, by a Quantity less than $\frac{1}{497}$ Part of the Diameter. But if the Proportion of 7 to 22, and 71 to 223 be reduced to a common Consequent, there will arise the Proportions of 1561 to 4906, and of 1562 to 4906.

Therefore the Circumference being suppos'd to be 4906 Parts, the Diameter less than the true will be 1561, the Diameter greater than the true 1562.

Both therefore differ from the true Diameter by a Quantity less than $\frac{1}{4906}$ Part of the Circumference.

The Proportion delivered by *Metius* is much more accurate than this of *Archimedes*. According to this,

The Diameter is 113.

The Circumference 355.

Amongst all Proportions consisting of small Numbers, none comes nearer to the true one; for from this, the Diameter being suppos'd of 10,000,000 Parts, the Circumference comes to be of 31,415,929, which differs from the true one only in the first Figure 9, and this by an excess but a little greater than two ten-millioneth Parts of the Diameter.

But more exact than both is that double Proportion of *Ludolphus a Ceulen*; the former of which consists of 21 Figures, and the latter of 36.

The Diameter
100,000,000,000,000,000,000.
The Circumf. greater than the true
314,159265,358979,323847.
The Circumf. less than the true
314,159265,358979,323846.

The Difference of both the Circumferences is one Particle of the Diameter denominated from a Number which consists of a Unity and 20 Cyphers; and consequently as well this as that differs from the true Circumference by a Quantity less than is the said small Part of the Diameter; to wit, one hundredth of a millioneth of a millioneth of a millioneth Part.

The Diameter
100000,000000,000000,000000,000000,000000.
The Circumf. greater than the true
314159,265358,979323,846264,338327,950289.
The Circumf. less than the true
314159,265358,979323,846264,338327,950288.

The Difference of both the Circumferences betwixt which is the true one, is that small Part of the Diameter, denominated from a Number which consists of Unity and 35 Cyphers; which small Part bears a less Proportion to the whole Diameter, than one little Grain of Sand doth to the whole Globe of the Earth. For the whole Globe of the Earth doth not consist of so many little Grains of Sand, as are the little Parts of the said Sort which are contain'd in the Diameter.

It is needless to go any further. Nevertheless you may proceed infinitely, if you be minded to continue Geometrical Reasoning, an expedite Method of which is delivered by *Snellius*.

2 [*The*

[*The Circumference being suppos'd of*
1,000000,000000,000000,000000,000000,000000,000000 *Parts.*
The Diameter will be as near as may be, of
0,318309,886183,790671,537767,526745,028724 *Parts.*]

Scholium.

THE most excellent Advantages of the Proportion now delivered, are these which follow.

The Invention of the Diameter from the Circumference.

SET the greater Term of one of the Proportions which have been now delivered in the first Place, the lesser in the Second, the Circumference in the Third; by these three Numbers let there be sought by the Golden Rule a Fourth Proportional. That is the Diameter sought.

As, if the Circumference of the greatest Circle of the Earth be suppos'd to contain 25000 *English* Miles of 5280 Feet each, and the Diameter be sought; the Terms will stand thus,

355——113——25000——7958

Multiply now the second by the third, and divide the Product by the first; and there will arise 7958 Miles for the Diameter of the Globe of the Earth.

The finding out of the Circumference from the Diameter.

LET the lesser Term of one of the Proportions above delivered be set in the first Place; the greater in the second; the known Diameter in the third: and by these three Numbers let there be sought a fourth Proportional, That will give the sought Circumference.

As if the Diameter of the Globe of the Earth be suppos'd to contain 7958 *English* Miles; and the Circuit is sought; the Terms will stand thus.

113——355——7958——25000.

Then multiply the second by the third, and divide the Product by the first; there will arise 25000 Miles for the Circumference of the Globe of the Earth.

How little this Circumference exceeds the true one was said above; to wit, by an Excess but a little greater than are two ten-millioneth Particles of the Earth's Diameter; that is, by 9 or 10 Feet. But if we use the *Ludolphin* Proportion, even the former, the Terms whereof consist of 21 Figures; there will be found a Circumference insensibly differing from the true, not only when the given Diameter is of 7958 Miles, such as is the Diameter of the Earth; but also altho the Diameter be suppos'd of 300 Millions of those Miles. For this being suppos'd, there will arise a Circumference differing from the true one by a Quantity about one hundred-millioneth Part of a Foot. But if to find out the Circumference of the Globe of the Earth, we make use of the Proportion of *Archimedes*, the difference of the two Circumferences, the one greater, the other less than the true one, will exceed 15 Miles. *Archimedes*'s Proportion therefore is not to be used but in small Measures; nay, it will always be expedient to use that of *Metius*, which both consists of small Terms, and is above a 1000 times more exact.

The measuring of a Circle.

THE Semidiameter multiplied by half the Circumference produceth the Area of the Circle; as is manifest from *Corol.* 1. *Prop.* 5. of this Book.

As if the Semidiameter of the Earth, which contains 3979 Miles, be multiplied by half its Circumference, to wit, by 12500, there will arise 49,737500 Miles square for the Area of the greatest Circle of the Earth, The Difference of the circular Area thus found from the true is had, if the Difference of half this found Circumference from the true half-Circumference be multiplied by the given Semidiameter; or the difference of this Semidiameter from the true, be multiplied by the given Semicircumference.

The Mensuration of Cylinders and Cones,

I Put this here, because it depends upon the Mensuration of a Circle, A Cylinder therefore, and any Prism what-

whatsoever is produced from the Altitude multiplied by the Base: A Cone and Pyramid from the third Part of the Altitude multiplied by the Base; for they are third Parts of Cylinders and Prisms, having the same Base and Altitude with them, by 10, and 7, *l.* 12.

Let the Base of a Cylinder or Cone be of 50 square Feet and the Height of 100 Feet. Multiply 100 by 50, and there arise 5000 cubick Feet for the Solidity of the Cylinder. Multiply the third Part of the Altitude 100, which is $33\frac{1}{3}$ by 50, there arise $1666\frac{2}{3}$ cubical Feet for the Solidity of the Cone.

PROP. VII. Theorem.

THE Circumferences of Circles have the same Proportion betwixt themselves which their Diameters have. *Fig.* 6, & 7. *l.* 12.

For the Circuits of like Polygons, which may be inscribed in a Circle without end, are always betwixt themselves as the Diameters A F and I C (by *Coroll. Pr.* 1. *l.* 12.) But these Circuits (by the third *Pr.* of this Book) end at length in the Circumference. Therefore their Circumferences also are betwixt themselves as their Diameters. *Q. E. D.*

PROP. VIII. Theorem.

THE Superficies of a Prism, as well that which is circumscrib'd about, as that which is inscrib'd in a Cylinder, is equal to a Rectangle whose Height is the Side of the Cylinder, but its Base equal to the Circuit of the Base of the Prism.

Part I. The Superficies of the circumscrib'd Prism touches the Cylinder according to the Lines E A, N F, &c. which are the Sides of the Cylinder; but these (because by the Hypothesis the Cylinder is a right one) are right to the Plane of the Base, and consequently right also (by *Defin.* 3. *l.* 11.) to the Lines C G, G M, &c. But they are also equal betwixt themselves. Therefore *Fig.* 3.

one Side of the Cylinder is the common Height of all the Rectangles C O, O M, M H, &c. Therefore the Superficies of the circumscribed Prism is equal (as is manifest from 1. *l.* 6.) to a Rectangle contain'd under the Circuit of the Base of the Prism, and the Side of the Prism or Cylinder.

Part II. The Reason of this is the same. For the Side of the Cylinder is again the common Altitude of the Rectangles B D I K, K I Q P, &c. which constitute the Superficies of the inscribed Prism.

PROP. IX. Theorem.

THE *Superficies of a regular Pyramid circumscrib'd about a right Cone, is equal to a Triangle, which hath for its Base the Circumference (F H L D) of the pyramidal Base, but its Height the Side of the Cone (B G).*

And the Superficies of a regular Pyramid inscribed in a right Cone, is equal to a Triangle, which hath for its Base the Circumference of the pyramidal Base, but for its Height the Perpendicular (B O) let down from the Top unto a Side of the Base.

Part I. Let there be drawn unto the Contacts, G, K, M, the right Lines B G, B K, B M. These will all be Sides of a right Cone, and consequently equal. And, because (by the Hypothesis) the Axis B A is perpendicular to the Plane of the Base F K D, the Plane also G B A (*per* 18. *l.* 11.) will be perpendicular to the Plane F K D. But H G (*per* 18. *l.* 3.) is perpendicular to A G, the common Section of the Planes F K D and G B A. Therefore H G (as is gathered from *Defin.* 4. *l.* 11.) is also perpendicular to the Plane G B A. And consequently is also perpendicular to B G. Therefore the Side G B of the Cone, is the Height of the Triangle F B H. In the same manner the Side of the Cone will be the Height of the rest H B L, L B D, &c. Therefore the Triangle comprehended under the Circumference F H L D and the Side of the Cone is equal to the Superficies

of a Pyramid circumscribed, without the Base. Which was the first Part.

II. The Demonstration of this Part is almost the same with that of the former.

PROP. X. Theorem.

THE *Superficies of a regular Prism circumscrib'd about a right Cylinder, ends* (Defin. 6. l. 12.) *in the Superficies of the Cylinder; and the Superficies of a Pyramid circumscrib'd about a right Cone, ends in the Superficies of the Cone.*

Part I. The Superficies of regular Prisms describ'd about, and inscrib'd in a Cylinder without end, will have at last a difference betwixt themselves less than any which can be given (*by* 8 *and* 3 *of this*.) Much more therefore will the Superficies of a circumscrib'd Prism differ from the Superficies of the Cylinder, which is in the middle between the inscribed and circumscribed Superficies, by a Difference less than any given one whatsoever; that is, (*Def.* 6. *l.* 12.) will end in the cylindrical Superficies, whilst it continually exceeds it less and less. *Fig.* 3.

Part II. This may be shewed in the same manner from the 9 and 3 of this. *Fig.* 4.

In the Figures there are only exhibited the Halves of the Cylinder and Cone, lest a Multitude of Lines should breed Confusion. But the Cylinder and Cone are to be conceiv'd in the Mind entire, and as having these circumscrib'd Prisms and Pyramids encompassing them. For thus it more clearly appears that plain Surfaces circumscribed are greater, according to the 3d *Axiom*.

A Lemma *to the following* Proposition.

LET AB, CD, EF, be proportional, and let KB be half AB, and EG double EF; KB, CD, EG, will also be proportional. *Fig.* 7.

The right Line KB is to AB as EF is to EG. Therefore the Rectangle KB, EG (*per* 16. *l.* 6.) is equal to the Rectangle AB, EF. But this (by 17. *l.* 6.) is

is equal to the Square of CD. Therefore also the Rectangle KB, EG, is equal to the Square of CD. Therefore (by 17. *l.* 6.) KB, CD, EG are proportional.

PROP. XI. Theorem.

Fig. 5, 6. A *Circle, whose Radius (GH) is a mean proportional betwixt the Side of a right Cylinder (BC) and the Diameter of the Base (BD) is equal to the cylindrical Superficies.*

Let the regular and consequently like Polygons, NM, RS, be understood to be circumscribed about the Circles ABN, GPH; and upon the Polygon NM let a Prism be conceiv'd to be erected, circumscribed about the Cylinder. Because BD, GH, BC are by the Hypothesis proportional, AD also (or AN), GH, and the double of BC will, by the *Lemma*, be proportional. Now the Triangle contain'd under AN, and the Circuit of the Polygon MN, is equal to the Polygon circumscribed NM (by the fourth of this Book): and the Rectangle under BC, or EF, and the same Circuit NM (that is, as is manifest from 41. *l.* 1. the Triangle under the Circuit NM, and the double of BC) is equal (by the 8th of this Book) to the Superficies of a Prism circumscrib'd about the Cylinder. But a Triangle under the Circuit NM and AN, is to the Triangle under the Circuit NM, and the double of BC (by 1. *l.* 6.) as AN is to the double of BC. Therefore the Polygon NM also is to the Superficies of a Prism circumscribed about a Cylinder, as AN is to the double of BC. But because I have already shew'd AN, GH, and the double of BC to be proportional, the Proportion of AN to the double of BC is (by *Defin.* 10. *l.* 5.) duplicate to the Proportion of AN to GH. Therefore the Polygon NM hath to the Superficies of the Prism a Proportion duplicate to the Proportion of AN to GH. But the Polygon NM hath also to the Polygon like to it GRQS a Proportion duplicate to the Proportion of AN to GH, as is easily gathered out of 1. *l.* 12. Therefore the Polygon NM hath the same Proportion to the Superficies of the Prism, which it hath to the Polygon GRQS; which consequently

consequently is equal to the Superficies of the Prism. In the same manner, I might shew that the prismatic Superficies, which are circumscriptible infinitely about the Cylinder, are always equal to the Polygons which may be circumscribed infinitely about the Circle G P H. Wherefore seeing both the prismatic Superficies (by the 10th of this) end in the Surface of the Cylinder, and the Polygons in the Circle G P H, (by the 3d of this); the Superficies of the Cylinder also will be equal to the Circle G P H. *Q. E. D.*

From this admirable Theorem, a Circle is presented which is equal to a cylindrical Superficies.

Corollaries.

THE Superficies of a right Cylinder is equal to a Rectangle contain'd under the Side (B C) and the Circumference of the Base. Fig. 5, 2.

The double of B C (as hath been shew'd above) is to G H, as G H is to B A, or A N; that is, (by the 7th of this) as the Circumference P is to the Circumference B N. Therefore the Triangle under the first, to wit, the double of B C, and the fourth, to wit, the Circumference B N, is equal to a Triangle under the second G H, and the third, to wit, the Circumference P, (as appears from *16. l. 6.*) But the Triangle under G H and the Circumference P, is (by the 5th of this) equal to the Circle G P H, that is, (by the 11th of this) to the cylindrical Superficies. Therefore also the Triangle under the double of B C and the Circumference B N, (that is, as appears from 41. *l.* 1. the Rectangle which is under B C and the Circumference B N) will be equal to the cylindrical Superficies. *Q. E. D.*

From this *Corollary* it is manifest, that the Properties of Rectangles are common to them with right cylindrical Superficies. Therefore let this be *Corollary* 2.

2. The cylindrical Superficies (B M, Q N) which are of the same Height, are betwixt themselves as the Diameters of their Bases (B F, Q R). Fig. 20, *l.* 12.

For the Rectangles under the Circumferences (C L, S E) and the equal right Lines F M, R N, to which (by

(by *Corol.* 1.) the cylindrical Superficies are equal, are betwixt themselves (by 1. *l.* 6.) as the Bases, to wit, the Circumferences C L, S E; that is, as the Diameters B F, Q R (by the 7th of this.)

3. The cylindrical Superficies (C I, A R) which have equal Bases, are betwixt themselves, as their Altitudes (T I, B R).

Fig. 23, 24. l. 12. For the Rectangles contain'd under the equal Circumferences G B, M Q, and the Sides T I, B R, to which (by *Corol.* 1.) the cylindrical Surfaces are equal, are betwixt themselves (by 1. *l.* 6.) as T I, B R.

Fig. 20, 21. l. 12. 4. Like cylindrical Surfaces (B M, R I) have betwixt themselves a Proportion duplicate to that which (B F, Q R,) the Diameters of the Bases have.

Seeing the Cylinders are suppos'd to be like, M F will be to I Q (by *Defin.* 4. *l.* 12.) as B F is to Q R; that is, (by the 7th of this) as the Circumference C L to the Circumference S E. Wherefore the Rectangles also which are contain'd under the Circumferences C L, S E, and the Sides M F, I Q, will be like; and consequently they will have betwixt themselves (by 20. *l.* 6.) a Proportion duplicate to that which M F hath to I Q; that is, B F to Q R. Therefore the cylindrical Surfaces also have the same.

The same Figure. 5. Cylindrical Surfaces (B M, R I,) have betwixt themselves a Proportion compounded of the Proportions of the Sides (F M, I Q,) and the Diameters of the Bases (B F, Q R,) as is manifest from 23. *l.* 6. and the 7th of this.

Fig. 24. 25. l. 12. 6. If cylindrical Surfaces (A R, F D) be equal; as the Diameter (A B) is to the Diameter (F N,) so reciprocally (by 14. *l.* 6.) the Altitude (F H) will be to the Altitude (B R); and conversly.

7. Lastly, from the same 1st *Corol.* is had the Measure of a cylindrical Superficies; to wit, if the Circumference of the Base be multiplied by the Altitude. As if the Altitude be of 20 Feet, the Circumference of the Base of 6; multiply 20 by 6, there arises 120 square Feet for the Cylindrical Superficies.

PROP.

PROP. XII. Theorem.

THE Superficies of a right Cylinder is to the Base (A B N) as the Side of the Cylinder (C B) is to (B O) the fourth Part of the Diameter of the Base.

Let G H be a mean Proportional betwixt B C the Height, and B D the Diameter of the Base, and consequently (by *Lemma* before *Prop.* 11. of this) a mean Proportional betwixt A N and the double of B C. The Circle G P H of the Radius G H is (by the 11th of this) equal to the curve cylindrical Superficies C D. But the Circle G P H hath to the Base of the Cylinder A B N a Proportion duplicate (by 2. *l.* 12.) to the Proportion of G H to A N; that is, the same which the double of B C hath to the Radius B A (by the Hypothesis, and *Def.* 10. *l.* 5.) that is, the same which B C hath to B O, the fourth Part of the Diameter. Therefore the cylindrical Superficies also is to the Base A B N as B C is to B O, the fourth Part of the Diameter. *Q. E. D.* *Fig.* 6, 5. of this.

Corollary.

THE Superficies of a Cylinder which hath its Sides equal to the Diameter of its Base, is fourfold of the Base. But if the Side be a fourth Part of the Diameter of the Base, the Superficies of the Cylinder will be equal to the Base. Both these are manifest from the Proposition.

PROP. XIII. Theorem.

A Circle whose Radius (O L) is a mean Proportional betwixt the Side (B C) of a right Cone, and the Radius of the Base (A C) is equal to the conical Superficies. *Fig.* 9, 8.

Let regular Polygons E F, I N, be understood to be circumscrib'd about the Circles A C G, O P L, and a Pyramid

Pyramid circumscrib'd about the Cone to be erected upon the Polygon E F.

Because, by the Hypothesis, A C, or A G, is to O L, as O L is to B C, the Proportion of A G to B C will (*Defin.* 10. *l.* 5.) be duplicate to the Proportion of A G to O L. But as A G is to B C, so is the Triangle under A G and the Circuit E F to the Triangle under B C and the same Circuit E F. Therefore the Proportion of the Triangle under A G and the Circuit E F, to the Triangle under B C, and the same Circuit, is also duplicate to the Proportion of A G to O L. But the Triangle under A G, and the Circuit E F is equal to the Polygon L F (by the 4th of this): And the Triangle under B C and the same Circuit E F (by the 9th of this) is equal to the Superficies of the circumscribed Pyramid. Therefore the Proportion of the Polygon E F to the Superficies of the Pyramid is also duplicate to the Proportion of A G to O L. But the Proportion of the Polygon E F to the Polygon I N, which is by the Construction like to it, is (*per* 1. *l.* 12.) also duplicate to the Proportion of A G to O L. Therefore the Polygon E F hath the same Proportion to the Superficies of the Pyramid, and to the Polygon I N, which consequently are equal. In the same manner I might shew that the Superficies of Pyramids, which may be circumscrib'd about a Cone infinitely more and more, are always equal to Polygons which may be circumscribed infinitely about the Circle O P L. Wherefore seeing both the Surfaces of Pyramids (by the 10th of this) do at last end in the Surface of the Cone, and Polygons (by the 3d of this) in the Circle O P L, the Superficies of the Cone and the Circle O P L, shall likewise be equal. *Q. E. D.*

From this excellent Theorem a Circle is found which is equal to a conical Surface.

Corollaries.

Fig. 9, 8. 1. THE Superficies of a right Cone is equal to a Triangle comprehended under the Side of the Cone (B C) and the Circumference of the Base (C G). I

Let O L the Radius be a mean Proportional betwixt A C and B C. Then because (*by the 7th of this*) the Circumference

Circumference CG is to the Circumference P as the Radius AG is to the Radius OL; that is, by the Hypothesis, as OL is to BC; the Triangle under the first, to wit, the Circumference CG and under the 4th BC (*as appears from* 16. *l.* 6.) will be equal to the Triangle under the second; to wit, the Circumference P, and the third OL; that is, (*by the* 5*th of this*) to the Circle OPL; that is, to the Conical Superficies (*by the* 13*th of this*) BCD. *Q. E. D.*

From this *Corollary* it appears that conical Surfaces have the same Properties with Triangles. And so it follows,

2. That the conical Superficies (BAF, QXR) having their Sides (BA, QX) equal, are betwixt themselves as the Diameters of their Bases, (BF, QR). And, *Fig.* 20, 21. *l.* 12.

3. Those which have equal Bases CFT, AZB, are betwixt themselves as their Sides (CF, AZ). And, *Fig.* 23, 24. *l.* 12.

4. Those conical Superficies (BAF, QZR) which are like, have betwixt themselves a Proportion duplicate to that which is betwixt the Diameters of their Bases. And, *Fig.* 20, 21. *l.* 12.

5. All conical Superficies whatsoever have betwixt themselves a Proportion which is compounded of the Proportions of the Sides (BA, QZ) and of the Diameters (BF, QR) which are in the Bases. And, The same Figure.

6. Those which are equal have their Sides and the Diameters of their Bases reciprocally proportional; and those which have them so, are equal.

All which is demonstrated from *Coroll.* 1. as above we deduced the *Corollaries* concerning the cylindrical Surface out of the first *Corollary* there.

7. Lastly, we may measure a right conical Surface, if we multiply the Side FC by half the Circumference of the Base. As if the Side be of 5 Feet, the Circumference of the Base of 20; multiply 5 by 10, and there will arise 50 square Feet for the conical Superficies. The Demonstration is manifest from the same first Corollary. *Fig.* 25. *l.* 12.

PROP-

PROP. XIV. Theorem.

Fig. 8, 9. of this. *THE Superficies of a right Cone is to the Base, as the Side (B C) is to (A C) the Radius of the Base.*

Between the Side BC and AC the Radius of the Base, let OL be a mean Proportional. Therefore the Proportion of BC to AC is duplicate to the Proportion of OL to AC; *(Defin. 10. l. 5.)* Now *(by the 13th of this)* a Circle of the Radius OL is equal to the conical Superficies CBD. But the Proportion of this to ACG the Base of the Cone is (by 2. *l.* 12.) duplicate to the Proportion of OL to AC; and consequently is the same with the Proportion of BC to AC. Therefore the Proportion of the conical Superficies CBD is to the Base ACG, as BC is to AC. *Q. E. D.*

Corollaries.

Fig. 27. *THE Superficies of a right Cone produc'd by an equilateral Triangle turn'd about the Perpendicular (K A) is double to the Base (Q T).*

For the Side KB is equal to BD, and consequently double to the half of it AB, which is the Radius of the Base.

Fig. 24. 2. *The Superficies of a Cone produc'd by a right angled equicrural Triangle (E B D) is to the Base, as in a Square the Diameter is to the Side.*

For the Perpendicular BA being drawn, the right Angle B (by 26. *l.* 1.) is bisected, and consequently ABD is half right. But ADB is also an half right Angle; (by *Coroll.*11. *pr.*32. *l.* 1.) Therefore DA, BA, are (by 6. *l.*1.) equal; and consequently BD is the Diameter of the Square AK, whereof AD is the Side. Now the same AD is the Semidiameter of the Base PT, seeing the Perpendicular AB *(by 26. l. 1.)* bisects ED.

E D. From which, and this 14th, the *Corollary* is manifest.

3. *The Superficies of a right Cylinder, (G K) is to the Superficies of a right Cone (G B N), as the Side of the Cylinder is to half the Side of the Cone.* Fig. 24.

For the Superficies of the Cone G B N is to the Base M I, as the Side B N is to Q N the Semidiameter of the Base *(by the 14th of this)*; that is, as half the Side B N is to the fourth Part of the Diameter G N. But the Base M I *(by the 12th of this)* is to the Superficies of the Cylinder G K, as the fourth Part of the Diameter is to N K, the Side of the Cylinder. By Equality of Proportion therefore the conical Superficies G B N is to the cylindrical Superficies G K, as half the Side of the Cone is to N K, the Side of the Cylinder. *Q. E. D.*

A Lemma *to what follows.*

IN a Triangle, as N P V, let there be drawn Q D parallel to N V. Fig. 10.

I say that the Rectangle under P N and N V is equal to the Rectangle under P Q, Q D, together with the Rectangle under N Q, and the two N V, Q D, put together.

Draw N A perpendicular to the Side N P, and equal to N V; and the Rectangle N O being compleated, let the Diameter P A be drawn. Then from Q let there be drawn Q E parallel to N A, which may cut P A in B. Thro' B let C F be drawn parallel to N P. Because A N is equal to N V, it is manifest that Q B also is equal to Q D, (from *Coroll.* 1. *p.* 4. *l.* 6.) Therefore the Rectangle O N is the Rectangle P N V, and F Q is P Q D. It remains that we prove that the Rectangles O B, E C, B N, are equal to the Rectangle under N Q, and the two N A, B Q; that is, to the Rectangle under N Q, and the two Lines N V, Q D. But that is manifest; for the Rectangle under N Q, and N A, Q B, is equal *(per* 1. *l.* 2.) to these three Rectangles; that under N Q and C A (that is, the Space E C,) and that under N Q and N C (that is, the Space B N,) and that under

under N Q and Q B, that is again the Space B N, and consequently the Space O B which (*per* 43. *l.* 1.) is equal to B N. The Proposition therefore is manifest.

PROP. XV. Theorem.

Fig. 11, 12. *IF a right Cone be cut by the Plane Q S B parallel to N Z O; I say, that the Circle G H M whose Radius G H is a Mean betwixt Part of the Side N Q, and Q D, N V (the Radius's of the Circles Q S B, N Z O) taken together; is equal to the conical Surface intercepted betwixt the parallel Circles Q S B, N Z O.*

Let G F be the Mean betwixt P N and N V. Likewise let G K be the Mean betwixt P Q and Q D; and let there be described the Circles G F L, G K T. This (*by the* 13*th of this*) will be equal to the conic Superficies Q P B, and the other to the Superficies N P O. The Rectangle P N V (by the *Lemma*) is equal to the Rectangle P Q D, together with the Rectangle under N Q and N V, Q D, taken together. But because (by the Construction) G F is a mean Proportional betwixt P N, N V; the Rectangle P N V is equal to the Square of G F (by 17. *l.* 6.) And because G K is (by the Construction) a Mean betwixt P Q, Q D, the Rectangle (by 17. *l.* 6.) P Q D is equal to the Square of G K: And because G H by the Hypothesis is a Mean betwixt Q N, and Q D, N V, taken together, the Rectangle (by 17. *l.* 6.) under Q N, and Q D, N V, taken together, is equal to the Square of G H. Therefore the Square of G F is also equal to the Square of G H, and to that of G K. Therefore seeing Circles are betwixt themselves (by 2. *l.* 12.) as the Squares of their Radius's, the Circle G L F will also be equal to the two Circles G K T, G H M taken together. But (*by the* 13*th of this*) the Circle G L F is equal to the conical Superficies N P O. Therefore the conical Superficies N P O is also equal to the two Circles G K T, and G H M. But Q P B one Part of the Superficies N P O is (*by the same*) equal to the Circle G K T. Therefore the remaining Part, which is comprehended betwixt the parallel Circles Z Z, S S, is equal to the Circle G H M. *Q. E. D.*

4 *A Lem-*

A Lemma to what follows.

RIght Lines (BH, CG) which in the Circle intercept equal Arches (BC, HG) are parallel. *Fig.* 13.

For let CH be drawn. Because the Arches BC, HG are by the Hypothesis equal, the alternate Angles also (by 29. *l.* 3.) BHC, GCH, will be equal. Therefore (by 28. *l.* 1.) BH, and CG are parallel. *Q. E. D.*

PROP. XVI. Theorem.

LET there be inscrib'd in a Circle a regular Figure of an even Number of Sides, and let it be equilateral; let EB be drawn from the Extremity of the Diameter unto B, the end of the Side next to the Diameter: and let the right Lines BH, CG, DF, join the Angles which are equally distant from A. *Fig.* 13.

I say that the Rectangle contain'd under the Diameter AE, and the Subtense EB, is equal to the Rectangle of one Side of the inscrib'd Figure AB, or BC, &c. *and of all the joining Lines BH, CG, DF, taken together.*

Draw CH, DG: Because BH, CG, DF intercept (by 26. *l.* 3.) equal Arches, BC, HG; CD, GF; these Lines (by the *Lemma*) will be parallel. By the same Argument BA, CH, DG, EF, are parallel. All the Triangles therefore (by 27, and 15. *l.* 1.) BAK, KHL, LCM, MGN, NDO, OFE, are equiangular. Therefore (by 4. *l.* 6.) as BK, is to KA, so is HK to KL; and as HK is to KL, so is CM to ML; and as CM to ML, so is GM to MN; and as GM is to MN, so is DO to ON; and as DO is to ON; so is FO to OE. Therefore (by 12. *l.* 5.) as one of the Antecedents, BK, is to one of the Consequents KA; so all the Antecedents BK, KH, CM, MG, DO, OF, (that is, all the joining Lines BH, CG, DF) are to all the Consequents AK, KL, LM, MN, NO, OE (that is, to the Diameter AE.) But (by 8. *l.* 6.) as BK is to AK, so is EB to BA. Therefore as all these together BH, CG,

DF are to AE, so is EB to BA. Therefore (by 16. *l.* 6.) the Rectangle under BA on one Part, and all the joining Lines BH, CG, DF, on the other, is equal to the Rectangle which is under AE and EB. *Q. E. D.*

PROP. XVII. Theorem.

Fig. 14. *LET there be inscrib'd in DAF a Segment of a Circle, whose Base DF is perpendicular to the Diameter AOE, a Figure equilateral, and of an even Number of Sides; and let there be drawn, as in the foregoing, the Line EB.*

I say, that the Rectangle comprehended under EB, and AO part of the Diameter, is equal to the Rectangle which is under one Side of the inscrib'd Figure, and all the joining Lines BH, CG, &c. *taken together with DO half the Base.*

The Demonstration is the same with that of the foregoing.

Lemma 1. *to what follows.*

Fig. 13. LET there be inscribed in the greatest Circle of a Sphere a regular Figure, which hath its Sides measured by the Number Four, and stands about the Axis AE; which Axis remaining unmov'd, let the Circle be turn'd round together with the Figure:

I say, that there will be inscrib'd in the Sphere a Body contained under conical Superficies.

It is manifest (see *Defin.* 2. *l.* 12.) that BA, HA, likewise DE, FE, describe entire Superficies of right Cones. Then because the Lines CB, GH, and GF, CD, being produced, do concur on both Sides in the same Point of the Diameter AE, which is in like manner to be drawn out, and cuts the joining Lines perpendicularly; it is also manifest that the said Lines CB, GH, &c. do describe Parts of right conical Surfaces, which are intercepted betwixt the parallel Circles, which the Tops of the Angles B, C, D, describe in the spherical Superficies.

Lemma 2.

Lemma 2.

LET DAF be the greatest Section of a Segment of a Sphere, whose Axis is A O. Let there be inscribed in this a Figure having all the Sides equal, the Base excepted, and let it be turn'd round about the Axis A O. Fig. 14.

I say, that a Body contain'd under conical Superficies will be inscrib'd in the spherical Segment.

This is proved as the foregoing *Lemma*.

PROP. XVIII. Theorem.

LET the same Things be supposed which were in the first Lemma; *and let the right Line (E B) be drawn from the Extremity of the Diameter unto the end of the Side next to the Diameter.* Fig. 13.

I say, that a Circle, the Square of whose Radius (I) is equal to the Rectangle A E B, contain'd under the Diameter A E, and the Subtense E B, is equal to all the conical Surfaces inscrib'd in the Sphere.

That is a Circle whose Radius (I) is a mean Proportional betwixt A E and E B.

Because the right Lines B H, C G, D F, are equal to the right Lines B K, CM, D O, taken twice; (by 1. *l.* 2.) the Rectangle under one Side of the Figure inscrib'd in the great Circle (to wit, under A B, or B C, or C D, or D E,) and under all the joining Lines together B H, C G, D F, is equal to the Rectangle under A B and B K, together with that which is under B C, and the compound of B K and C M, together with that which is under C D and the Compound of C M and D O, together with that which is under D E and D O; for so each of the Lines B K, C M and D O, are taken twice. But (*by the 16th of this*) the Rectangle under A B and all the joining Lines B H, C G, D F, taken together, is equal to the Rectangle A E B; that is, (by the Hypothesis) to the Square of I. Therefore the Square of I

is equal to the Rectangles under A B and B K, and under B C and the Compound of B K and C M, under C D and the Compound of C M and D O, and under D E and D O. Now let P be a mean Proportional betwixt A B and B K; and Q a Mean betwixt B C and the Compound of B K and C M; and R a Mean betwixt C D and the Compound of C M, D O; & a Mean betwixt D E and D O. The Squares therefore of P, Q, R, S, (by 17. *l.* 6.) are equal to the abovesaid Rectangles. Wherefore seeing I have already shew'd the Square of I to be equal to the same Rectangles, it must also be equal to the Squares of P, Q, R, S, together. Seeing therefore (by 2. *l.* 12.) Circles are betwixt themselves as the Squares of their Radius's; the Circle described by the Radius I, will also be equal to all the Circles together whose Radius's are P, Q, R, S, (as is manifest from 22. *l.* 6.) But the Circles of the Radius's P and S, are (*by the 13th of this*) equal to the conical Superficies which the Sides A B, E D, have produc'd; forasmuch as P is a mean Proportional betwixt A B the Side of the Cone, and B K the Radius of the Base; and S is a mean Proportional betwixt E D and D O; and the Circle of the Radius Q is (*by the 15th of this*) equal to that Segment of a conical Superficies, which is intercepted betwixt the two parallel Circles of the Diameters C G, B H, because Q is a Mean betwixt B C and the Compound of B K, C M: And for the same Cause the Circle of the Radius R is equal to a Segment of a conical Surface, which is intercepted betwixt the parallel Circles of the Diameters C G, D F. Therefore the Circle described from the Radius I, is equal to all the conical Surfaces inscribed in the Sphere taken together. *Q. E. D.*

PROP. XIX. Theorem.

Fig. 14. *LET the same Things be supposed which were in the* 2*d* Lemma, *and let the right Line E B be drawn from the Extremity of the Diameter (A E) to the end of A B the Side next to the Diameter.*

I

I say, that a Circle whose Radius is a mean Proportional betwixt (E B) and (A O) the Axis of the Segment, is equal to all the conical Superficies inscribed in the spherical Segment D A F.

The Demonstration is altogether the same with that of the foregoing; only for *Prop.* 16. let *Prop.* 17. be cited.

PROP. XX. Theorem.

Conical Superficies inscrib'd in a Sphere, do at length end in the Superficies of the Sphere. Fig. 15.

Let there be given a Superficies as small as you will, as X. It is manifest that within the spherical Superficies A C E G, there may be given some other Concentrical thereto, which falls short of this by a Quantity less than X. Let A C E G, D P L M, be the greatest Circles of both, as cut with a Plane thro' the Centre. Draw the Diameter A D E, and in D let N Q touch the lesser Circle. If the Arch A E be bisected in C, and again the Remainder be bisected, and so on, there will be left at last the Arch A B (as is manifest of it self:) less than the Arch A N. If to this Arch the right Line A B be subtended, it is manifest that it will not reach to the Circumference P D M L, and that it will be a Side of an equilateral Figure of an even Number of Sides, inscrib'd in the Circle C A G E, no Side whereof reacheth unto the Circumference P D M L. Wherefore if all be turn'd round about the Diameter A E, it is manifest that there will be inscribed in the exterior spherical Surface conical Surfaces, which include the spherical Surface, which is concentrical to the other, and consequently (by *Axiom* 3. of this) are greater. Because therefore the spherical Surface D P L M falls short of the spherical Surface A C E G, by a Quantity less than the given one X; much more will the conical Surfaces fall short of the said spherical Surface A C E G by a Quantity less than the given one X, and (by *Defin.* 6. *l.* 12.) consequently will end in the Superficies A C E G. *Q. E. D.*

PROP. XXI. Theorem.

Fig. 17. *Conical Superficies inscrib'd in a spherical Segment D A F, end in the spherical Superficies of the Segment it self.*

This may be demonstrated by the same Reasoning as the foregoing was.

PROP. XXII. Theorem.

Fig. 16. *IT was demonstrated,* Prop. 18. *that a Circle whose Radius is a mean Proportional betwixt the Diameter A E and the right Line E B, which is drawn from the Extremity of the Diameter unto the end of the Side A B next to the Diameter, is equal to all the conic Superficies inscrib'd in the Sphere.*

I say, that this Circle (see Def. 6. l. 12.*) ends at length in a Circle, whose Radius is A E the Diameter of the Sphere.*

For if more and more Sides be infinitely inscribed in the greatest Circle (which then being turn'd round about A E produce conical Superficies); it is manifest that the Side A B becomes at length less than any given right Line, and consequently that the Subtense E B approaches to the Diameter A E, within a Distance less also than any given one; from whence it comes to pass that the Difference of those A E, B E, becomes likewise less than any given one. Therefore much more shall the mean Proportional betwixt A E, B E, which is always greater than B E, differ from A E at length by a Defect less than any given one. Therefore the Circle also whose Semidiameter is a Mean betwixt A E and B E, will at length differ from a Circle whose Semidiameter is A E, by a Defect less than any given one whatsoever; that is, will end *(Def. 6. l. 12.)* in it. *Q. E. D.*

This which is clear enough of it self, there is no need to demonstrate more operosely.

PROP.

PROP. XXIII. Theorem.

IT was demonstrated, Prop. 19. *that a Circle whose Radius is a mean Proportional betwixt E B and the Axis of the Segment A O, is equal to all the conical Superficies inscrib'd in the spherical Portion D A F.* Fig. 17.

I say, that this Circle ends in a Circle whose Radius is the right Line A D, drawn from the Vertex of the Segment unto the Circumference of the Circle D Q F N, which is the Base of the Segment.

For because it now appears from the foregoing Demonstration that E B doth at length end in A E; it will also be manifest that the mean Proportional betwixt E B and A O doth at length end in the mean Proportional betwixt A E and A O, that is, (by *Coroll.* 2. *p.* 8. *l.* 6.) in A D it self. It is therefore manifest that the Circle also whose Radius is a mean Proportional betwixt E B and A O, doth end in the Circle of the Radius A D. *Q. E. D.*

A Lemma *to the following* Proposition.

IF the Diameter of one Circle be double to the Diameter of another, the one Circle will be fourfold to the other.

This is manifest from *Prop.* 2. *l.* 12. and *Defin.* 10. *l.* 5.

PROP. XXIV. Theorem.

THE Superficies of every Sphere is fourfold of the greatest Circle of the same Sphere. Fig. 16.

This most noble Theorem of *Archimedes* we shall from what goes before expeditiously demonstrate in this manner.

Let an ordinate Figure, the Sides whereof are measured by the Number Four, be understood to be inscrib'd in

in the greateſt Circle of a Sphere about the Diameter A E. Let this Figure be turn'd round about A E, and ſo produce conical Surfaces inſcrib'd in the ſpherical Surface, and let E B be drawn. It hath been demonſtrated above (18. *of this*) that all conical Surfaces inſcribed in a Sphere are equal to the Circle, the Square of the Radius whereof is equal to the Rectangle A E B, that is, whoſe Radius is a mean Proportional betwixt A E and E B. And this will always happen, altho the Inſcription be infinitely continued. Wherefore ſeeing the inſcrib'd conical Surfaces (*by* 20. *of this*) will at length end in the ſpherical Surface, and the Circle whoſe Radius is a Mean betwixt A E and E B, will at length end (*by* 22. *of this*) in the Circle whoſe Radius is A E; the ſpherical Surface it ſelf alſo will be equal to the Circle of the Radius A E, that is, (by the foregoing *Lemma*) to four Times the greateſt Circle A C E G. *Q. E. D.*

He that ſhall read *Archimedes*, will find that the Way here uſed in demonſtrating this moſt noble Theorem, is much ſhorter and clearer than that of *Archimedes*.

Corollary.

FRom this admirable Theorem, whereby *Archimedes* hath purchas'd to himſelf an immortal Name amongſt the Geometricians, a Circle equal to a ſpherical Surface is obtain'd; that, to wit, whoſe Semidiameter is the Diameter of the Sphere, or whoſe Diameter is double to the Sphere's Diameter.

Scholium.

WE are now well provided for the meaſuring of a ſpherical Surface, the chief amongſt all Curve ones. And it is perform'd theſe two Ways.

1. Let the greateſt Circle of the Sphere be meaſured (according to *Schol. Prop.* 6. of this) and let it be multiplied by 4. As, if the greateſt Circle of the Earth be found to contain 49,737,500 Square Miles, then according to this, 198,950,000 ſquare Miles are contain'd in the whole ſpherical Surface.

2. The

2. The Diameter of a Sphere multiplied by the Circumference of the greatest Circle gives you the spherical Superficies. According to which, if the Earth's Diameter consist of 7958 Miles, and consequently the Circumference of the greatest Circle consists of 25,000, the whole spherical Surface will be in the same Miles 198,950,000; for $7958 \times 25,000 = 198,950,000$.

The Demonstration appears from *Coroll.* 1. *Prop.* 5. of this; for a Rectangle under the Diameter of a Sphere, and the Circumference of the greatest Circle, is according to that *Corollary* fourfold of the greatest Circle.

PROP. XXV. Theorem.

THE Surface of any spherical Portion whatever (as D A F) is equal to a Circle, whose Radius is the right Line (A D) drawn from the Vertex of the Portion to the Circumference of the Circle (D Q F N) which is the Basis of the Segment. Fig. 17.

Let a Figure equilateral and of an even Number of Sides, the Base being set aside, be imagin'd to be inscrib'd in the greatest Section of the Segment about the Axis A O; this Figure being turn'd round about A O will inscribe conical Surfaces in the Portion. Let the right Line E B be drawn also as above (*in* 18 *and* 19 *of this.*) All the conical Surfaces now inscribed are equal (*by the* 19*th of this*) to the Circle whose Radius is a mean Proportional betwixt E B and the Axis of the Segment A O. And this will always happen if the Inscription be infinitely continued. Wherefore seeing both the conical Surfaces inscrib'd in the Segment end at length (*by* 21 *of this*) in the spherical Surface of the Segment, and the Circle whose Radius is a Mean betwixt E B and A O ends (*by* 23.) in the Circle of the Radius A D; the spherical Surface of the Portion also D A F will be equal to the Circle of the Radius A D. *Q. E. D.*

This is another of the more noble Inventions of *Archimedes*, which, as the former, we have demonstrated in a much shorter and clearer Way than he did.

PROP.

PROP. XXVI. Theorem.

Fig. 18. *THE Superficies of a right Cylinder circumſcrib'd about a Sphere (as the Cylinder HPSV) is equal to the Surface of the Sphere.*

And if a Cylinder and Sphere be cut by Planes perpendicular to the Axis (B G); each Segment of the Cylindrical Surface will be equal to each Segment of the Spherical Surface.

Part I. Becauſe the Side HP of the Cylinder is *(by the Hypotheſis)* equal to PS the Diameter of the Baſe; the Cylindrical Surface HS will be (by *Coroll. p.* 12. of this) fourfold of the Baſe; that is, of the greateſt Circle of the Sphere inſcrib'd in the Cylinder; of which ſeeing *(by 24th of this)* the ſpherical Surface it ſelf is alſo fourfold, this will be equal to the Cylindrical Surface. *Q. E. D.*

Part II. Let the right Lines BO, GO, be drawn. Becauſe the Angle BOG *(by* 31. *l.* 3.*)* is right, as being the Angle in the Semicircle, and OC falls perpendicular from it upon BG; BO (by *Corol.* 2. *p.* 8. *l.* 6.*)* will be a mean Proportional betwixt GB and BC, that is, betwixt IT and HI. Therefore the Circle of the Radius BO *(by* 11. *of this)* will be equal to the Cylindrical Surface HT. But the ſame Circle is alſo *(by the foregoing)* equal to the Segment of the ſpherical Surface OBK. Therefore the Cylindrical Surface HT and the ſpherical OBK are equal.

Then becauſe it is ſhew'd in the ſame manner that the Cylindrical Surface HX is equal to the ſpherical QBR, the remaining Cylindrical Surface IX will be equal to the remaining ſpherical Surface QOKR, which is intercepted betwixt two parallel Circles.

And from theſe the Propoſition is manifeſt of all Segments.

[Coroll. *Hence the Superficies of a Cylinder circumſcrib'd about a Sphere is double to the Baſes.*]

PROP.

PROP. XXVII. Theorem.

THE Segments of a spherical Surface divided by parallel Circles have that Proportion amongst themselves, which the Segments (BC, CD, DA, AE, EF, FG) of that Diameter (BG) which is perpendicular to the parallel Circles have amongst themselves. Fig. 18.

It follows from the foregoing. For by that the Segments of the spherical Surface OBK, QOKR, MQRN, &c. are equal to the Cylindrical HT, IX, LN, &c. But these (by 13. *l.* 12.) have the same Proportion betwixt themselves which the Segments of the Axis BC, CD, DA, &c. have. Therefore those also have the same Proportion. *Q. E. D.*

Scholium.

FRom hence the Proportion of Zones and Climates betwixt themselves becomes known. For they are to one another as the Segments of the Axis, which are known from the Table of Sines.

From the same also we learn to measure the Segments of a spherical Surface. For because both the whole Surface of the Sphere is known from *Schol. Prop.* 24. and the Proportion of the Segments, the same as that of the Parts of the Axis, is also given; it is manifest that each of the Segments become known.

Now both the foregoing, and all the rest of the Theorems which follow, are altogether singular and admirable, and well worthy that those who are studious of Geometry should give all Diligence to understand them.

A Lemma *to the following.*

IF a Plane (QN) touch a Sphere in (O), a right Line (AO) from the Centre to the Contact is perpendicular to the Plane. Fig. 19.

Let

Let QN the touching Plane and the Sphere be cut thro' the Contact with two Planes, which in the Sphere may produce the Circles OG, OD, but in the Plane QN the right Lines CO, IO, which ſhall touch the Circles in O. Therefore by 18. *l.* 3. AO is perpendicular to both IO and CO, and conſequently by 4. *l.* 11. perpendicular to the Plane QN. *Q. E. D.*

PROP. XXVIII. Theorem.

Fig. 20, 22, 21.

Every Sphere is equal to a Cone (ZO) whoſe Altitude (KO) is equal to the Radius of the Sphere; and the Baſe (Z) equal to the Superficies of the Sphere.

Let ſome Polyedral Body be underſtood to be circumſcribed about the Sphere, and let the ſolid Angles thereof be cut off by new Planes touching the Sphere. Which being done, there will ariſe another Polyedral Body containing the Sphere, but leſs than the former, and conſiſting of more Angles, and having a Surface compounded of more tangent Planes in Number, but leſs in Magnitude. If the ſolid Angles of this Polyedrum be again cut off by new tangent Planes, and the Angles of the third Polyedrum thence ariſing likewiſe, and ſo on for ever; it will come to paſs at length that both the Polyedrum will exceed the Sphere by a ſolid leſs than any given one whatſoever; and the Surface thereof compounded of tangent Planes (which, as I ſaid, are endleſly leſs in Magnitude, and more in Number than they were before) will exceed the ſpherical Surface alſo by a Plane leſs than any given one whatever. Both which Things, altho they might be demonſtrated, yet becauſe they are of themſelves manifeſt enough, I ſhall, for Brevity-ſake, take for granted. Theſe Things being thus ſtated, we proceed.

The Polyedrum now deſcrib'd is compounded of Pyramids, the common Top whereof is the Centre of the Sphere, and the Baſes are tangent Planes, which conſtitute the Surface of the Polyedrum. And becauſe the right Lines drawn from the Centre A unto the Contacts of each of the Planes, are (*by the foregoing Lemma*) perpendicular to each of the Planes; therefore the Height of all the Pyramids, whereof the Polyedrum conſiſts, will

will be equal; to wit, AB the Radius of the Sphere: If therefore the Plane X be suppofed equal to the Surface of the Polyedrum it felf, and upon it there be erected a Pyramid at the Height MN, which is alfo equal to the Radius of the Sphere; it is manifeft (by *6. l. 12.*) that all the abovefaid Pyramids, that is, the whole Polyedrum, are equal to the Pyramid XN. After the fame manner all the reft of the Polyedrums containing the Sphere, which from the perpetual Abfciffion of the folid Angles will arife one after another infinitely, are always equal to the Pyramids (reprefented by XN), the Altitudes whereof MN are the Radius of the Sphere; but the Bafes (X) equal to the Surfaces of Polyedrums encompaffing the Sphere. Wherefore, feeing at length both the Polyedrums (as I faid above) do end in a Sphere, and the Pyramids, (XN) as I will fhew by and by, do end in the Cone ZO; the Sphere alfo will be equal to the Cone. *Q. E. D.*

But that the Pyramids XN end in a Cone, I thus fhew. The Surfaces of Polyedrums end in the Surface of the Sphere, as it was taken for granted above. But the Bafes X of the Pyramids XN are always fuppos'd equal to the Surfaces of the Polyedrums; and Z, the Bafe of the Cone ZO, is by the Hypothefis equal to the Surface of the Sphere; therefore the Bafes X alfo will end in the Bafe Z; and confequently feeing the Pyramids XN be to the Cone, which by the Hypothefis is of equal Height, (by *Corol. Prop.* 11. *l.* 12.) as the Bafe X is to the Bafe Z, the Pyramids alfo will end in the Cone.

The Demonftration of this Propofition and the following is altogether diverfe from that which *Archimedes* made ufe of, which indeed is very fubtile and ingenious, but prolix and difficult; to which there are premis'd two Pofitions that are manifeft, and eleven Propofitions, befides others not a few, on which they depend. But the Theorem it felf, as propounded by *Archimedes*, is thus: Every Sphere is fourfold of a Cone, which hath a Bafe equal to the greateft Circle of the Sphere, and its Altitude equal to the Radius.

Scholium.

FRom this noble Theorem is deduc'd the Menfuration of the moft noble of folid Figures. For if the Sixth

Part

Part of the Diameter, or the third Part of the Semidiameter, be multiplied by the Surface of the Sphere, already known by *Schol. Prop.* 24. there will arise the Solidity of the Sphere.

Suppose the Superficies of the Earth be found to contain 198,950,000 square Miles, and let the third Part of the Semidiameter consist of 1326 such Miles. Multiply the two Numbers together, the Product 263807,700000 will be the Number of the cubic Miles of the Earth's Solidity.

For seeing a Sphere (by this *Prop.*) is equal to a Cone whose Altitude is the Radius of the Sphere, and its Base the Surface of the same Sphere, and the Solidity of the Cone (by *Schol. Prop.* 6. of this) is produc'd from the third Part of the Altitude (that is, of the Radius of the Sphere) multiplied by the Base (that is, the Surface of the Sphere,) the Sphere's Solidity also is obtain'd from the 3d Part of the Radius multiplied into the Superficies.

PROP. XXIX. Theorem.

Fig. 23. *EVery Sector of a Sphere is equal to a Cone whose Altitude is the Radius of the Sphere, and the Base the spherical Superficies of the Sector.*

First, let the Sector A E C G be less than an Hemisphere. Let a right-lin'd polyedral Body be understood to be circumscrib'd about the Sector. Now if all the remaining Ratiocination be carried on after the same manner as was done in the foregoing, the Thing sought will be concluded in the same manner. This Thing alone will require to be shew'd, upon which indeed the whole Reasoning depends; to wit, that the Superficies of the Polyedrum, which is compounded of Planes on every Side, touching the Surface of the Sphere E C G, is greater than the Surface E C G. Which is done thus. Let another equal and like Surface be conceiv'd to be set to the Surface E C G, encompass'd with touching Planes in the very same manner as the other is. Then will (by *Axiom* 3. of this) the whole Surface compounded of Planes, be greater than the whole spherical Surface. Therefore half the Surface compounded of Planes will also be greater than half the spherical Surface E C G.

Then

Then let the Sector A E B G be greater than an Hemisphere. Both Sectors taken together are (by the foregoing) equal to a Cone whose Height is the Radius of the Sphere, its Basis the whole Superficies; that is, (by 11. *l.* 12.) to two Cones which have the same Height, but have their Bases equal to the Segments of the spherical Superficies E C G, E B G. But one of the Sectors A E C G, that which is less than an Hemisphere, is by Part 1. equal to a Cone, whose Altitude is the Radius, and its Base the Surface E C G. Therefore the other Sector A E B G is equal to the other Cone whose Height is the Radius, and its Base the remaining spherical Surface E B G. *Q. E. D.*

Corollary.

SEeing (by 25. of this) the Superficies E C G is equal to the Circle of the Radius C G, and the Superficies E B G equal to the Circle of the Radius B G; the Sectors A E C G, and A E B G, will be equal to Cones whose Altitude is the Radius of the Sphere, and their Bases Circles of the Radius's C G, and B G.

Scholium.

FRom these Things is deduc'd the measuring both of Sectors and Segments of Spheres; of Sectors (as appears from *Schol. Prop.* 6. of this) if the third Part of the Radius be multiplied by the spherical Surface of the Sectors, which is already known from *Schol. Prop.* 27. or by the Circle of the Radius C G or B G; and of Segments, if the Cone E A G be measured, and be taken away from the Sector, if it be less than an Hemisphere; but added thereto, if it be greater. Fig. 23.

The Segment (M Q R N) which lies betwixt two Circles, whether parallel or not parallel, is measur'd; if the Segments Q B R and M B N already known, be substracted one out of the other. Fig. 18.

PROP. XXX. Theorem.

AN Hemisphere (E O B D) is double to a Cone (E B D) which hath the same Base and Altitude with it self. Fig. 24.

Q The

The Cone whose Basis is the hemispherical Superficies E O B D, and its Altitude the Radius A B, is to the Cone E B D (by 11. *l.* 12.) as Base is to Base; that is, as the hemispherical Surface E O B D is to the greatest Circle P T. Therefore seeing the hemispherical Superficies E O B D is double to the greatest Circle (by 24. of this), the Cone also which hath the Superficies E O B D for its Base, and the Radius A B for its Altitude, is double to the Cone E B D. But (by 28. of this) the Hemisphere is equal to a Cone which hath the Radius for its Altitude, and the hemispherical Superficies for its Base. Therefore the Hemisphere is also double to the Cone E B D. *Q. E. D.*

PROP. XXXI. Theorem.

Fig. 25. *LET a Sphere be divided into two Segments I L B G, I S K G, by the Plane I Q G T which doth not pass thro' the Centre A; and let the Diameter B O K be perpendicular to the cutting Plane.*

As the Altitude O B of the Segment I L B G, is to the Radius of the Sphere A B: So let O K, the Altitude of the other Segment, be made to another Line K N.

In like manner, As O K, the Altitude of the Segment I S K G, is to the Radius A K or A B, So let the Altitude O B of the other Segment be made to another Line B D. Which Things being suppos'd, I say,

1. The Cones I N G and I D G, whose Altitudes are O N, O D, and I Q G T their common Base, are equal to the spherical Segments.

2. There is the same Proportion of the Segments as there is of the right Lines D Q, N O.

3. The Segment I S K G is to the greatest Cone I K G inscrib'd in it, as N O is to K O; and the Segment I L B G is to the greatest Cone I B G inscrib'd in it, as D O is to B O.

Part

Part I. Let the Sphere and Cones be cut by a Plane thro' the Diameter BK. There will be produced in the Sphere the greatest Circle BLKG; and in the Cones the Triangles BIG, IKG. And because BOK the Diameter is (by the Hypothesis) perpendicular to the Circle QT, IOB (by *Def.* 3. *l.* 11.) will be a right Angle. The Angle BIK in the Semicircle is also a right one (by 31. *l.* 3.) Because therefore in the Triangle BIK, there is drawn from the right Angle, IO perpendicular to the Base BK; BI will be to IO, as (by 8. *l.* 6.) BK to KI. Therefore the duplicate Proportion of BI to IO is equal to the duplicate Proportion of BK to KI; that is, (because BK, KI, KO [by *Corol.* 2. *Pr.* 8. *l.* 6.] are three Proportionals) equal to the Proportion of BK to KO.

Then because OB is (by the Hypothesis) to BD, as OK is to the Radius AB; by Inversion it will be always thus, DB is to BO, as AB to OK; and by Permutation thus, DB is to BA, as BO to OK; and by Compounding thus, DA is to BA, as BK is to OK. Because therefore I have already shew'd the Proportion of BK to OK to be duplicate to the Proportion of BI to IO, and consequently (by 2. *l.* 12) equal to the Proportion betwixt the Circles describ'd by the Radius's BI, IO; DA will also be to BA, as the Circle of the Radius BI, to the Circle of the Radius IO. Therefore the Cone under the Altitude DA, and for the Base, the Circle of the Radius IO, that is, the Circle QT, is equal to the Cone under the Altitude BA, (by 15. *l.* 12.) which hath for its Base the Circle of the Radius BI; that is, (by *Corol. Pr.* 29. of this) the spherical Sector AIBG. Wherefore if the same Cone IAG be added as well to the Sector AIBG, as to the Cone under DA, and the Circle QT, the Wholes will be equal; to wit, the spherical Segment ILBG will be equal to two Cones, whereof one is that which is under the Base QT and the Altitude DA, and the other IAG is under the same Base QT, and the Altitude OA. But these two Cones (by 14. *l.* 12.) make up the Cone IDG. Therefore the Segment ILBG will be equal to the Cone IDG. *Q.E.D.*

By the same Reasoning, the Segment ISKG will be equal to the Cone ING, with this only Change,

that the Cone I A G, which before was added, be now taken away.

Part II. This is manifest from the first. For the Cones I D G and I N G are betwixt themselves (by *p.* 14. *l.* 12.) as are D O and N O. Therefore the Segments also I L B G, I S K G, equal to those Cones, are betwixt themselves, as the right Lines, D O, N O.

Part III. This likewise is manifest from the first. For the Cone I D G is to the Cone I B G, (by the same) as D O is to B O. Therefore the Segment also I L B G, which is equal to the Cone I D G, is to the Cone I B G, as D O is to B O.

Scholium.

FRom the first Part of this Proposition there arises another Way of measuring spherical Segments, and that a very easy one; if, to wit, the Cones I D G, I N G, be measured; which will be done if the third Parts of the right Lines D O, N O, be drawn into the Circle Q T.

PROP. XXXII. Theorem.

Fig. 24. *A Right Cylinder (G K) is both in Solidity and the whole Superficies to the Sphere about which it is circumscrib'd as 3 to 2.*

Let B Q be the common Axis of the Sphere and Cylinder, and E B D the greatest Cone inscrib'd in the Hemisphere E O B D. Because the Cylinder E K (half of G K) is (by 10. *l.* 12.) triple to the Cone E B D, while the Hemisphere is double to the same Cone (by 30 of this), it is manifest that the Cylinder E K is to the Hemisphere as 3 to 2. Therefore also the whole Cylinder G K is to the whole Sphere Q E B D, as 3 to 2. Which was the first Part.

Then because the Side of the Cylinder K N is equal to G N the Diameter of the Base, its Superficies without the Bases will be fourfold (by *Corol. Pr.* 12. of this) of the Base M I, and consequently taken together with the Bases, that is, the whole Superficies of the Cylinder, will be sixfold of the Base M I, which is equal to the greatest Circle of the Sphere. But the Superficies of the Sphere is fourfold of that greatest Circle. Therefore the

the whole Superficies of the Cylinder G K is to the Superficies of the Sphere, as 6 to 4, or as 3 to 2. Which was the other Part.

Therefore a Cylinder is both in Solidity and the whole Superficies to the Sphere, about which it is circumscrib'd, as 3 to 2. *Q. E. D.*

Scholium.

IT is an Argument what a great Value *Archimedes* puts upon this Theorem, that he would have a Sphere inscrib'd in a Cylinder set upon his Tomb. And perhaps amongst so many other famous Discoveries, this chiefly and above all others pleas'd him, for this Reason, to wit, because there was one and the same rational Proportion both of Bodies, and of the Surfaces which contain them. We have demonstrated a like Identity of Affections betwixt Rings, and the Surfaces of Rings, in the 4th Book of our Cylindricks and Annularies, *Prop.* 13, 14, 15. And another famous Example of the same hath also offer'd it self to me in the Sphere it self. For I have found, that like as a Sphere is to a right Cylinder which encompasseth it (which will necessarily be equilateral) as 2 is to 3, and this both in respect of Solidity and Surface; so likewise the Sphere hath to an equilateral Cone encompassing it, that Proportion which 4 hath to 9; and this both in regard of Solidity and Superficies. From which this also follows, That the sesquialteral Proportion found by *Archimedes* in the Sphere and Cylinder, is continued in three Solids, a Sphere, Cylinder, and equilateral Cone. The Demonstration of both which Things, with some other Theorems of my own, in which the wonderful Nature of the Sphere will more appear, I shall subjoin in the thirteen following Propositions.

PROP. XXXIII. Theorem.

THE Superficies of a Sphere is double to the Superficies of a square Cylinder inscrib'd in the same Sphere. *Fig.* 26.

Let A K L D be the Square inscrib'd in the greatest Circle of a Sphere, from which turn'd round, there is

describ'd a square Cylinder; and let AL be drawn as a Diameter common to the Square and Sphere. Because the Square of AL is (by 47. *l.* 1.) equal to the equal Squares of AK, KL, it will be double to one AK. Therefore also the Circle of the Diameter AL, is (by 2. *l.* 12.) double to the Circle, whose Diameter is AK; to wit, to the Circle CN. But the Superficies of the Sphere is (by 24. of this) fourfold to the Circle whose Diameter is AL; for that is the greatest Circle of the Sphere, seeing AL is the Diameter of the Sphere. Therefore the Superficies of the Sphere is eightfold of the Circle CN. But because LK, KA (by the Hypothesis) are equal, the cylindrical Superficies ACL is (by *Corol. Pr.* 12. of this) quadruple of the Circle CN. Therefore since the Superficies of the Sphere is eightfold of the same Circle, it will be double to the cylindrical Superficies. *Q. E. D.*

PROP. XXXIV. Theorem.

Fig. 26. THE *Superficies of a Sphere hath that Proportion to the whole Superficies of a square Cylinder inscrib'd in it, which* 4 *hath to* 3.

Let the same Things be suppos'd which were in the foregoing Demonstration. Because by the Hypothesis LK the Side of the Cylinder, and AK the Diameter of the Base thereof are equal, the cylindrical Superficies CL will be quadruple (by *Corol. Pr.* 12. of this) to the Base CN, and consequently the whole Superficies of the Cylinder is to both Bases CN and SL, as 6 is to 2. But the Superficies of the Sphere is to both Bases together CN, SL, as 8 is to 2, seeing in the foregoing it was shew'd that it is to one Base as 8 to 1. Therefore the Superficies of the Sphere is to the cylindrical Superficies CL as 8 is to 6, or 4 to 3. *Q. E. D.*

Corollary.

THE whole Superficies of a right Cylinder describ'd about a Sphere, is to the whole Superficies of an equilateral Cylinder inscrib'd, as 2 is to 1. For the Circumscrib'd is to the spheric Superficies as 12 is to 8 (by

23.

32. of this) But the Spheric is to the Inscrib'd as 8 is to 6, by this present Proposition. Therefore the Circumscrib'd is to the Inscrib'd as 12 is to 6, or 2 to 1.

PROP. XXXV. Theorem.

THE Superficies of any spherical Portion whatever (as ILBG) hath the same Proportion to the Superficies of the greatest inscribed Cone, which (BG) the Side of the Cone hath to (GO) the Radius of the Base. Fig. 26, & 25.

Because (by 25. of this) the Superficies of the Portion ILBG is equal to the Circle of the Radius BG; the Proportion thereof to QT, that is, to the Base of it self and of the Cone, will be duplicate to the Proportion (by 2. *l.* 12.) of BG to GO; that is, (by 14. of this) of the Proportion of the conical Superficies IBG, to the same Base QT. Therefore it is manifest (by *Def.* 10. *l.* 5.) that the Superficies ILBG is to the conical Superficies IBG, as the same conical Superficies IBG is to the Base QT. Wherefore seeing the conical Superficies IBG, is to the Base QT, as BG (by 14. of this) is to GO, the Superficies of the Portion will also be to the conical Superficies IBG inscrib'd in it, as BG is to GO, *Q. E. D.*

PROP. XXXVI. Theorem.

THE Superficies of the Hemisphere (EOBD) hath that Proportion to (EBD) the Superficies of the greatest right inscribed Cone, which in a Square the Diameter hath to a Side; and that Proportion to the Superficies of a like Cone circumscribed, as the Side in a Square hath to the Diameter. Fig. 24.

I. The Demonstration of the first Part is manifest from the foregoing. For the Superficies of any Portion whatever, and consequently of the Hemisphere, EOBD, is to the conical Superficies inscrib'd, as BD is to DA. But BADK is a Square, whose Diameter is BD and the Side DA.

Part II. Let EBC be half of the Square circumscrib'd about the Circle (whose Centre is O); which EBC being turn'd about the Axis OB, let there from thence Fig. 6. l. 4.

be produc'd a Cone circumscribed about the Hemisphere. Now because the Square E C is (by 47. *l.* 1.) double to the Square E B or G I, the Circle of the Diameter E C also is (by 2. *l.* 12.) double to the Circle whose Diameter is G I, that is, to the Circle H G D I. But (by 24. of this) the Superficies of the Hemisphere included in the Cone E B C is double to the same Circle. Therefore the Circle of the Diameter E C is equal to the hemispherical Surface. Wherefore seeing the conical Superficies E B C is (by 14. of this) to the Circle of the Diameter E C, to wit, to its own Base, as the Side B E is to E O the Radius of the Base; it will be also to the hemispherical Superficies inscribed in it, as B E is to E O; that is, as the Diameter in a Square is to a Side. *Q. E. D.*

PROP. XXXVII. Theorem.

The same Figure with Fig. 13. l. 5.

A Sphere hath the same Proportion to a square conical Rhombus circumscribed about it, both in respect of the Solidity and Surface, which in a Square the Side hath to the Diameter.

Let the Square E B C F be circumscrib'd about H G D I, the greatest Circle of a Sphere, from which Square as turn'd round about the Axis B F, let a conical Rhombus encompassing the Sphere be produc'd.

As E B a Side of the Square (see *Fig.* 6. *l.* 4.) is to the Diameter E C, even so let S be made to R; (see *Fig.* 13. *l.* 5.) and let this Proportion be continued thro' four Terms, S, R, Q, O; the Proportion then of S to O will be triplicate to the Proportion of S to R; that is, of E B to E C, and the Proportion of Q to R will be duplicate to the Proportion of O to Q, or of R to S; that is, of E C to E B; and consequently (by 20. *l.* 6.) O is to R as the Square of E C is to that of E B; from whence (by *Schol. Pr.* 6. and 7. *l.* 4.) O is double to R. These Things being thus settled, let the Sphere E B C F be understood to be circumscribed about the conical Rhombus. Thus the Sphere H G D I will be to the Sphere E B C F (by 18. *l.* 12.) in the triplicate Proportion of the Diameter G I or E B to the Diameter E C; that is, (as I have already shew'd) it will be as S to O.

See def. 10. l. 5.

But

But the Sphere E B C F is to the conical Rhombus inscrib'd in it (by 30. of this) as 2 is to 1; that is, (as I have shew'd above) as O is to R. Therefore by Equality of Proportion, the Sphere H G D I is to the same Rhombus which is describ'd about it, as S is to R; that is, as in a Square the Side E B is to the Diameter E C. Which was the first Part. Then from the second Part of the foregoing, it appears that the Superficies of the Hemisphere is to the Superficies of the Cone E B C, and consequently the Superficies of the whole Sphere is to the Superficies of the whole Rhombus E B C F, as in a Square the Side is to the Diameter. Therefore the Sphere as well in Solidity as in Superficies is to the square Rhombus E B C F, as in a Square the Side is to the Diameter. *Q. E. D.*

PROP. XXXVIII. Theorem.

THE *Superficies of the Portion* (B G K D) *which contains an equilateral Cone* (B K D) *is double to the Superficies of the same Cone.* *Fig.* 27.

This is manifest from 35. For the Superficies of the Portion B G K D is to the inscrib'd conic Superficies (by 35. of this) as B K is to B A. But because the Cone B K D is suppos'd to be equilateral, K B is equal to B D, and consequently double to B A. Therefore the Superficies B G K D is also double to the inscribed conical Superficies B K D. *Q. E. D.*

PROP. XXXIX. Theorem.

THE *Superficies of a Sphere is to the whole Superficies of an equilateral Cone inscrib'd in it, as 16 to 9.* *Fig.* 27.

Let Z be the Center of the Sphere, and B K D the equilateral Cone inscribed, and K Z A O the Ax[illegible]mon to the Sphere and Cone. If the Sphere an[illegible]e be cut thro' this, there will be produced in the [illegible]ere the greatest Circle O B K D, and in the Cone th[illegible] equilateral Triangle B K D, one Side whereof B A D will be the Diameter of the Basis of the Cone Q T. And because

cause the Axis of the Cone K A is perpendicular to the Base QT, BAK (*Def.* 3. *l.* 11.) will be a right Angle. Therefore the Square of BA is equal to the Rectangle KAO. (*Corol.* 1. *Pr.* 17. *l.* 6.) Now because the Side of the equilateral Triangle cuts off (*Corol.* 5. *Pr.* 15. *l.* 4.) a 4th Part of the Axis AO, the Rectangle KAO, that is, the Square of BA, will be triple to the Square of AO (by 1. *l.* 6.) Wherefore seeing the Square of the Radius ZO is (*Corol.* 3. *Pr.* 4. *l.* 2.) quadruple of the Square of AO, the Square of the Radius ZO will be to the Square of the Radius BA, as 4 is to 3. Therefore the Circle OBKD is also (by 2. *l.* 12.) to the Circle QT, as 4 is to 3. Therefore four Circles OBKD, that is (by 24. of this) the whole spherical Superficies DG is to the Circle QT, as 16 is to 3. But (*Corol.* 1. *Pr.* 14. of this) the Superficies of the equilateral Cone BKD is to the Circle QT, to wit, its own Base, as 2 is to 1; and consequently the whole Superficies of the Cone BKD, including its Base, is to the Base, to wit the Circle QT, as 3 is to 1, or 9 to 3. Therefore seeing I have shew'd that the Superficies of a Sphere is to the same Circle, as 16 is to 3, the Superficies of the Sphere DG will be to the whole Superficies of the equilateral Cone, as 16 is to 9. *Q. E. D.*

Or otherwise thus:

BEcause (by *Corol.* 5. *Pr.* 15. *l.* 4.) the Side BD of the equilateral Triangle cuts off a 4th Part of the Axis AO, the spherical Superficies BOD will be a 4th Part by 27. of this, and consequently the Superficies BGKD, three 4th Parts of the Superficies of the whole Sphere. Wherefore if the whole Superficies be suppos'd to be 16, the Superficies BGKD will be 12. But (by the foregoing) the Superficies BGKD is double to the conical Superficies BKD, and consequently is to it, as 12 to 6. Therefore the whole Superficies of the Sphere is to the conical BKD, as 16 is to 6. Then because the Superficies of the Cone BKD (as being equilateral) is (by *Corol.* 1. *Pr.* 14. of this) double to the Base QT, it is manifest that the conical Superficies BKD (to wit, without the Base) is to the whole Superficies of the Cone, as 2 is to 3; that is, as 6 to 9. Therefore by equality of Proportion the whole Superficies of the

Sphere

Sphere is to the whole Superficies of the equilateral Cone inscrib'd, as 16 to 9. *Q. E. D.*

PROP. XL. Theorem.

THE Superficies of a Sphere bears that Proportion to the whole Superficies of an equilateral Cone circumscrib'd about it; which 4 doth to 9. Fig. 28.

Let there be circumscrib'd about the greatest Circle of a Sphere B P M, the equilateral Triangle D O F; by which, as turn'd round about the Axis O A B, let there be produc'd an equilateral Cone, circumscrib'd about the Sphere. And let there also be circumscrib'd about the equilateral Triangle D O F the Circle N D L O F, which, as is manifest, is concentrical to the former; and let the Axis O A B be produc'd to N. Because B N is a 4th Part of the Axis O N, (as is manifest from *Corol.* 5. *Pr.* 15. *l.* 4.) O N is double to B K. Wherefore the Proportion betwixt Circles being duplicate (by 2. *l.* 12.) of the Proportion of the Diameters, the Circle B P M will be to the Circle N D L O F, as 1 to 4. But it hath already been shew'd in the first foregoing Demonstration, that the Circle N D L O F is to the Circle Q T, the Base of the equilateral Cone inscrib'd in the Sphere F L, as 4 is to 3. Therefore by equality of Proportion the Circle B P M is to the Circle Q T, as 1 is to 3. But the whole Surface of the Cone D O F is (by *Cor.* 1. *Pr.* 14. of this) triple to Q T. Therefore the whole Superficies of the Cone is ninefold of the Circle B P M. Wherefore seeing the Superficies of the Sphere T P is quadruple (by 24. of this) of the same Circle B P M, the whole Superficies of the equilateral Cone D O F is to the Superficies of the Sphere to which it is circumscrib'd, as 9 is to 4. *Q. E. D.*

Coroll. 1. *From this Demonstration it is manifest that the Axis B O of an equilateral Cone circumscrib'd about a Sphere, is one and a half of the Diameter of the Sphere B K, or as 3 to 2.*

2. *That Q T the Base of the Cone D O F is also one and an half of both Bases of the Cylinder circumscrib'd about the same Sphere. For Q T is to B P M, as 3 to 1. Therefore Q T is to B P M twice, as 3 is to 2.*

3. *That*

3. That the Superficies of the Cone DOF is one and an half of the Superficies of the equilateral Cylinder circumscrib'd about the same Sphere. For That † *is double to QT, while this is quadruple to BPM* *. *Therefore the conical Superficies will be to the Cylindrical, as twice 3 to four times 1; that is, as 6 to 4, or as 3 to 2.*]

† Per Corol. 2. p. 14. of this.
* 24, and 26. of this.

PROP. XLI. Theorem.

Fig. 28.

THE whole Superficies of an equilateral Cone circumscrib'd about a Sphere, is quadruple to the whole Superficies of a Cone inscribed in the same Sphere.

By the foregoing the whole Superficies of the equilateral Cone DOF circumscrib'd, is to the Superficies of the Sphere, as 9 to 4; and the Superficies of the Sphere is the whole Superficies of the inscribed Cone SKT, as 16 to 9 (by 39. of this.) Therefore by Permutation of Equality of Proportion, the whole Superficies of the circumscribed equilateral Cone is to the whole Superficies of the equilateral inscrib'd, as 16 is to 4, or as 4 to 1. *Q. E. D.*

PROP. XLII. Theorem.

Fig. 29.

A Sphere hath that Proportion to BKC an equilateral Cone inscribed in it, which 32 hath to 9.

Let the Sphere and Cone be cut by a Plane passing thro' the common Axis KO, producing in the Sphere the greatest Circle OFKI, and in the Cone the equilateral Triangle BKC. Then a Plane being drawn thro' the Centre A perpendicular to OK, let the Hemisphere FGKI be cut off, in which let the greatest Cone FKI be understood to be inscribed. Now because (by *Cor.* 5. *p.* 15. *l.* 4.) the Side BC of the equilateral Triangle cuts off OP a 4th part of the Axis OK, PK will be to AK, as 3 to 2, that is, as 9 to 6. But the Base QT is to the Circle OFKI, that is, to the Base ND, as 3 to 4, that is, as 6 to 8, as appears from what was demonstrated *pr.* 39. Wherefore seeing the Proportion of the Cone BKC to the Cone FKI is (by *Schol.* 2. *pr.* 15. *l.* 12.) compounded of the Proportion of the Altitude PK to the Altitude AK (that is, of the Proportion of 9 to 6) and

and of the Proportion of the Base QT to the Base ND (that is, of the Proportion of 6 to 8) the Cone BKC will be to the Cone FKI, as 9 to 8. Wherefore seeing (by 30. of this) the Sphere CG is quadruple of the Cone FKI, the equilateral Cone BKC will be to the Sphere CG, as 9 to 32. *Q. E. D.*

PROP. XLIII. Theorem.

AN equilateral Cone circumscrib'd about a Sphere, is eightfold of an equilateral Cone inscrib'd in the same Sphere. Fig. 28.

Let SKT and DOF be the equilateral Cones inscrib'd and circumscrib'd, and let OKB be the common Axis. Then let as well both the Cones as the Sphere be cut by a Plane passing thro' the Axis; their Sections will be two equilateral Triangles, and the greatest Circle BPM. About the Triangle DOF likewise let there be understood to be describ'd the Circle NDOF, and let the Axis OKB be produc'd unto N. Now because the Side DF of the equilateral Triangle doth (by *Corol.* 5. *pr.* 15. *l.* 4.) cut off NB a 4th Part of the Axis ON, it is manifest that ON is double to BK. In like manner, because the Side ST of the other equilateral Triangle cuts off BC a 4th Part of the Axis BK, NO will be to BO, as BK is to CK; and by changing, as NO is to BK, so is BO to CK. But NO is double to BK. Therefore BO is likewise double to CK. Therefore because of the Similitude of the Triangles, DOF, SKT, DF and ST also, to wit, the Diameters of the conical Bases, will (by 4. *l.* 6.) be in a double Proportion betwixt themselves. Wherefore seeing the Cones DOF, SKT, be like, and consequently (by 12. *l.* 12.) their Proportion is triplicate to the Proportion of the Diameters DF and ST, which is that of 2 to 1, the Cone DOF will be to the Cone SKT, as 8 to 1. *Q. E. D.*

PROP. XLIV. Theorem.

A Sphere hath the same Proportion both in respect of Solidity and Surface to the equilateral Cone DOF circumscrib'd about it, which 4 hath to 9. Fig. 28.

The

The Sphere T P is (by 42. of this) to the equilateral Cone S K T inscrib'd in it, as 32 is to 9. But (by the foregoing) S K T the equilateral Cone inscrib'd is to D O F the equilateral Cone circumscribed, as 1 is to 8, that is, 9 to 72. Therefore by equality of Proportion the Sphere T P is to D O F the equilateral Cone circumscrib'd, as 32 is to 72, that is, as 4 to 9. But in *Prop.* 40. we demonstrated that the Superficies of a Sphere is to the whole Superficies of an equilateral Cone circumscribed, as 4 is to 9. Therefore a Sphere both in Solidity and Superficies is to an equilateral Cone circumscrib'd about it, as 4 is to 9. *Q. E. D.*

That therefore which *Archimedes* was surpriz'd at in a Sphere and Cylinder encompassing it, we have also now demonstrated in a Sphere and an equilateral Cone encompassing it, to wit, that there is the same rational Proportion of the Solidities betwixt themselves, which there is of the Surfaces. For as he found that the Sphere is to the Cylinder about it as well in Solidity as Superficies, as 2 to 3; so we have now taught, that the Sphere is in respect both of Solidity and Surface to an equilateral Cone encompassing it, as 4 to 9.

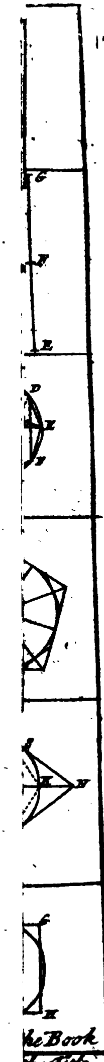
But from hence we shall without much labour demonstrate that the very Proportion, to wit, the sesquialteral, which *Archimedes* shew'd to be betwixt the Sphere and Cylinder, is continued by the equilateral Cone circumscrib'd both in the Solidity and Superficies; and so we shall put an End to the present Work.

PROP. XLV. Theorem.

See the Figure prefixed to this Treatise.

An equilateral Cone circumscrib'd about a Sphere, and a right Cylinder in like manner circumscrib'd about the same Sphere, and the same Sphere it self, continue the same Proportion; to wit, the sesquialteral, as well in respect of the Solidity as of the whole Superficies.

For by 32. of this Book, the right Cylinder G K encompassing the Sphere, is to the Sphere, as well in respect of Solidity as of the whole Superficies, as 3 is to 2, or as 6 to 4. But by the foregoing the equilateral Cone B A D circumscrib'd about the Sphere, is to the Sphere in both the said Respects, as 9 is to 4. Therefore the same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same Cone is to the Cylinder, both in respect of Solidity and Surface, as 9 is to 6. Wherefore these three Bodies, a Cone, Cylinder and Sphere, are betwixt themselves, as the Numbers 9, 6, 4, and consequently continue the sesquialteral Proportion. *Q. E. D.*

[*PROP.* *XLVI.*

THE same sesquialteral Proportion holds betwixt an equilateral Cone and Cylinder circumscrib'd about the same Sphere, in respect of their whole Surfaces, their simple Surfaces, their Solidities, Altitudes and Bases.

This Proposition is manifest as to the whole Surfaces and Solidities from the foregoing; as to the simple Surfaces, from Coroll. 3. Pr. 40. *of this; as to their Altitudes and Bases, from* Coroll. 1, *and* 2, *of the same* 40*th Proposition.*]

FINIS.